AU–18

SPACE HANDBOOK

AIR UNIVERSITY

MAXWELL AIR FORCE BASE, ALABAMA

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MATHEMATICAL SIGNS AND SYMBOLS

$\pm$ plus or minus, positive or negative

$\neq$ is not equal to

$\equiv$ is identical to

$\approx$ approximately equal to

$>$ greater than

$\geqq$ greater than or equal to

$<$ less than

$\leqq$ less than or equal to

$\backsim$ similar to

$\propto$ varies as, proportional to

$\rightarrow$ approaches as a limit

∞ infinity

$\therefore$ therefore

$\sqrt{\ }$ square root

$\sqrt[n]{\ }$ nth root

a^n nth power of "a"

a^{-n} reciprocal of nth power of a $= \left(\frac{1}{a^n}\right)$

log, $\log_{10}$ common logarithm

ln, $\log_e$ natural logarithm

n^o n degrees

n' n minutes; n feet

n'' n seconds; n inches

$f(x)$ function of x

Δx increment of x

dx differential of x

∂x partial differential of x

Σ summation of

$\int$ symbol for integration

GREEK ALPHABET

Alpha	A	α	Iota	I	ι	Rho	P	ρ
Beta	B	β	Kappa	K	κ	Sigma	Σ	σ
Gamma	Γ	γ	Lambda	Λ	λ	Tau	T	τ
Delta	Δ	δ	Mu	M	μ	Upsilon	Υ	υ
Epsilon	E	ε	Nu	N	ν	Phi	Φ	φ
Zeta	Z	ζ	Xi	Ξ	ξ	Chi	X	χ
Eta	H	η	Omicron	O	o	Psi	Ψ	ψ
Theta	Θ	θ	Pi	Π	π	Omega	Ω	ω

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MAXWELL AIR FORCE BASE, ALABAMA

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CHAPTER 2
ORBITAL MECHANICS

THE STUDY of trajectories and orbits of vehicles in space is not a new science but is the application of the concepts of celestial mechanics to space vehicles. Celestial mechanics, which is mainly concerned with the determination of trajectories and orbits in space, has been of interest to man for a long time. When the orbiting bodies are man-made (rather than celestial), the topic is generally known as orbital mechanics.

The early Greeks postulated a fixed earth with the planets and other celestial bodies moving around the earth, a geocentric universe. About 300 B. C., Aristarchus of Samos suggested that the sun was fixed and that the planets, including the earth, were in circular orbits around the sun. Because Aristarchus' ideas were too revolutionary for his day and age, they were rejected, and the geocentric theory continued to be the accepted theory. In the second century A.D., Ptolemy amplified the geocentric theory by explaining the apparent motion of the planets by a "wheel inside a wheel" arrangement. According to this theory, the planets revolve about imaginary planets, which in turn revolve around the earth. It is surprising to note that, even though Ptolemy considered the system as geocentric, his calculations of the distance to the moon were in error by only 2%. Finally, in the year 1543, some 1800 years after Aristarchus had proposed a heliocentric (sun-centered) system, a Polish monk named Copernicus published his *De Revolutionibus Orbium Coelestium,* which again proposed the heliocentric theory. This work represented an advance, but there were still some inaccuracies in the theory. For example, Copernicus thought that the orbital paths of all planets were circles and that the centers of the circles were displaced from the center of the sun.

The next step in the field of celestial mechanics was a giant one made by a German astronomer, Johannes Kepler (1571–1630). After analyzing the data from his own observations and those of the Danish astronomer Tycho Brahe, Kepler stated his three laws of planetary motion.

A contemporary of Kepler's, named Galileo, proposed some new ideas and conducted experiments, the results of which finally caused acceptance of the heliocentric theory. Some of Galileo's ideas were expanded and improved by Newton and became the foundation for Newton's three laws of motion. Newton's laws of motion, with his law of universal gravitation, made it possible to prove mathematically that Kepler's laws of planetary motion are valid.

Kepler's and Newton's work brought celestial mechanics to its modern state of development, and the major improvements since the days of Newton have been mainly in mathematical techniques, which make orbital calculations easier.

Because the computation of orbits and trajectories is the basis for predicting and controlling the motion of all bodies in space, this chapter describes the fundamental principles of orbital mechanics upon which these computations are based. It also shows how these principles apply to the orbits and trajectories used in space operations.

MOTION OF BODIES IN ORBIT

Bodies in space move in accordance with defined physical laws. Analysis of orbital paths is accomplished by applying these laws to specific cases. Orbital motion is different from motion on the surface of the earth; however, many concepts and terms are transferable, and similar logic can be applied in both cases. An understanding of simplified linear and angular motion will permit a more thorough appreciation of a satellite's path in space.

Linear Motion

Bodies in space are observed to be continuously in motion because they are in different positions at different times. In describing motion, it is important to use a reference system. Otherwise, misunderstanding and inaccuracies are likely to result. For example, a passenger on an airliner may say that the stewardess moves up the aisle at a rate of about 5 ft per sec, but, to the man on the ground, the stewardess moves at a rate of the aircraft's velocity plus 5 ft per sec. The man in the air and the man on the ground are not using the same reference system. For the present, the matter of a reference system will be simplified by first describing movement along a straight line, or what is called rectilinear motion.

Rectilinear motion can be described in terms of speed, time, and distance. Speed is the distance traveled in a unit of time, or the time rate of change of distance. An object has uniform speed when it moves over equal distances in equal periods of time. Speed does not, however, completely describe motion.

Motion is more adequately described if a direction as well as a speed is given. A speedometer tells how fast an automobile is going. If a direction is associated with speed, the motion is now described as a velocity. A velocity has both a magnitude (speed) and a direction, and it is therefore a vector quantity.

Uniform speed in a straight line is not the same as uniform speed along a curve. If a body has uniform motion along a straight line for a given time, then average velocity is represented by the equation $v = \frac{s_f - s_o}{t_f - t_o}$. In the equation, s_o is the initial position, s_f is the final position, t_o is the initial time, and t_f is the final time; or more simply, the velocity is the change in position divided by the change in time. The units of velocity are distance divided by time, such as ft per sec or knots (nautical miles per hour).* Since velocity is a vector quantity, it may be treated mathematically or graphically as a vector.

If velocity is not constant from point to point (i.e., if either direction or speed is changed), there is acceleration. Acceleration, which is also a vector quantity,

* The nautical mile (NM) is one minute of a great circle. In this course, use the conversion that 1 NM = 6,080 feet = 1.15 statute miles.

is the time rate of change of velocity. The simplest type of acceleration is one in which the motion is always in the same direction and the velocity changes equal amounts in equal lengths of time. If this occurs, the acceleration is constant, and the motion can be described as being uniformly accelerated.

The equation $a_{av} = \dfrac{v_f - v_o}{t_f - t_o}$ defines the average acceleration, over the specified time interval. A good example of a constant acceleration is that of a free-falling body in a vacuum near the surface of the earth. This acceleration has been measured as approximately 32.2 ft per sec per sec, or 32.2 ft per sec^2. It is usually given the symbol g. Since an acceleration is a change in velocity over a period of time, its units are ft/sec^2, or more generally, a length over a time squared. Actually, constant acceleration rarely exists, but the concepts of constant acceleration can be adapted to situations where the acceleration is not constant.

The following three equations are useful in the solutions of problems involving linear motion:

$$(1)\ s = v_o t + \frac{at^2}{2}$$

$$(2)\ v_f = v_o + at$$

$$(3)\ 2as = v_f^2 - v_o^2$$

where s is linear displacement, v_o is initial linear velocity, v_f is final linear velocity, a is constant linear acceleration, and t is the time interval.

Angular Motion

If a particle moves along the circumference of a circle with a constant tangential speed, the particle is in uniform circular motion. Since velocity signifies both speed and direction, however, the velocity is constantly changing because the direction of motion is constantly changing. Now, acceleration is defined as the time rate of change of velocity. Since the velocity in uniform circular motion is changing, there must be an acceleration. If this acceleration acted in the direction of motion, that is, the tangential direction, the magnitude of the velocity (the speed) would change. But, since the original statement assumed that the speed was constant, the acceleration in the tangential direction must be equal to zero. Therefore, any acceleration that exists must be perpendicular to the tangential direction, or in other words, any acceleration must be in the radial direction (along the radius).

Average speed is equal to the distance traveled divided by the elapsed time. For uniform circular motion, the distance in one lap around the circle is $2\pi r$, which is covered in one period (P). Period is the time required to make one trip around the circumference of the circle. Therefore, the tangential speed $v_t = \dfrac{2\pi r}{P}$. In uniform circular motion, the particle stays the same distance from the center, therefore, radial speed, $v_r = 0$. It has already been shown that $a_t = 0$; and it will be shown in the next section that $a_r = \dfrac{4\pi^2 r}{P^2} = \dfrac{v_t^2}{r}$.

Translatory motion is concerned with linear displacement, s; velocity, v; and acceleration, a. Angular motion uses an analogous set of quantities called angular displacement, θ; angular velocity, ω; and angular acceleration, α.

In describing angular motion, it is convenient to think of it in terms of the rotation of a radius arm (r), as shown in Figure 1. The radius arm initially coincided with the polar axis, but at some time later (t seconds) it was positioned as shown.

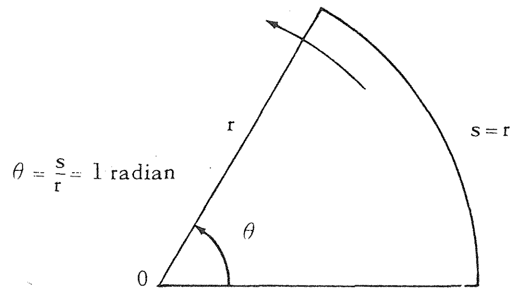


Figure 1. Position of radius arm as rotated one radian (57.3 degrees) from the starting point.

Angular displacement (θ) is measured in degrees or radians. A radian is the angle at the center of a circular arc which subtends an arc length equal to the radius length. If the length of s equaled the length of r, θ would be equal to one radian, or 57.3°. The central angle of a complete circle is 360° or 2π radians ($2\pi = 6.28$).

The following equations for angular motion are analogous to those studied earlier for rectilinear motion:

$$\begin{aligned}
\theta &= \frac{s}{r}\text{ radians} \\
\omega_{av} &= \frac{\theta_f - \theta_o}{t_f - t_o}\text{ rad/sec} \\
\alpha_{av} &= \frac{\omega_f - \omega_o}{t_f - t_o}\text{ rad/sec}^2 \\
\omega_f &= \omega_o + \alpha t \\
\theta &= \omega_o + \frac{\alpha t^2}{2} \\
2\alpha\theta &= \omega_f^2 - \omega_o^2
\end{aligned}$$

In the equations, θ_f is final angular position; θ_o is initial angular position; s is linear displacement (arc length); r is the radius; ω is average angular speed; ω_f is final angular speed; ω_o is initial angular speed; t_f is final time; t_o is initial time; and α is constant angular acceleration.

If a body is rotating about a center on a radius r, the tangential linear quantities are related to the angular quantities by the following formulas [where θ, ω, and α are in radians]:

$$\begin{aligned}
s &= r\theta \\
v_t &= r\omega \\
a_t &= r\alpha
\end{aligned}$$

Principles of the Calculus Applied to Astronautics

Computations in the calculus are based upon the idea of a limit of a variable. According to the formal definition, *the variable x is said to approach the constant 1 as a limit when the successive values of x are such that the absolute value of the difference x — 1 ultimately becomes and remains less than any preassigned positive number, however small.*

An example will make the definition easier to understand. The area of a regular polygon inscribed in a circle approaches the area of the circle as a limit as the number of sides of the polygon approaches infinity (Fig. 2).

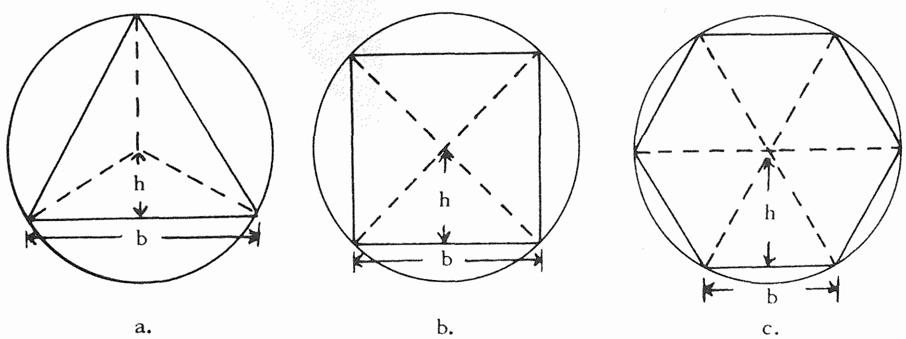


Figure 2. Increase in the number of sides of a regular polygon inscribed in a circle.

The area of a triangle is 1/2 bh. In general, if there are n sides to a polygon, the polygon is made up of n triangles as shown in Figure 2. Therefore, the area of the polygon is 1/2 nbh. As the number of sides (n) approaches infinity as a limit, the product nb approaches the circumference of the circle (c). Also, as n approaches infinity, the value of h approaches the radius (r) as a limit.

$$\lim_{n \to \infty} 1/2\ nbh = \frac{cr}{2}$$

This is read, "The limit of 1/2 nbh as n approaches infinity is equal to $\frac{cr}{2}$."

$$\text{But } c = 2\pi r$$

$$\therefore \lim_{n \to \infty} \text{area of the polygon} = \lim_{n \to \infty} \frac{nbh}{2} = \frac{(2\pi r)r}{2}$$

$$\text{and } \frac{(2\pi r)r}{2} = \pi r^2 = \text{area of the circle}$$

Now, an increment is the difference in two values of a variable. In the example above, the increase in area when the inscribed polygon increases the number of sides by one is an increment of area; that is, the area of an inscribed square minus the area of an inscribed triangle is an increment of area. An increment is written as Δx which is read "delta x," and does not mean Δ multiplied by x.

In the previous section the radial acceleration for uniform circular motion was given as $a_r = \frac{v_t^2}{r}$. With the concepts of an increment and a limit, the value for radial acceleration can be determined mathematically. In Figure 3, an increment of arc has been expanded to permit closer examination. The length r is the distance from the center of the circle to the circumference. The horizontal distance $v_t \Delta t$ is

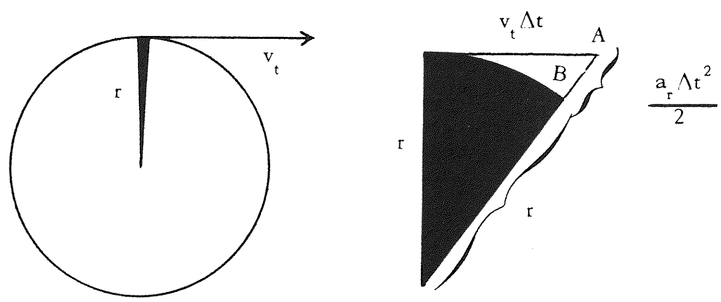


Figure 3. An increment of an arc (left) and the increment expanded (right) to show change in velocity.

the distance a body in uniform motion with a velocity v_t would move in the time Δt. However, at the completion of the increment of time the body is not at point A but at point B, because this is uniform circular motion. The distance from A to B is equal to $v_r \Delta t + \frac{a_r \Delta t^2}{2}$ where the subscript r refers to radial. However, for uniform circular motion $v_r = 0$. Therefore, the distance AB is equal to $\frac{a_r \Delta t^2}{2}$. Now, applying the Pythagorean theorem to the triangle,

$$r^2 + (v_t \Delta t)^2 = \left[r + \frac{a_r \Delta t^2}{2} \right]^2$$

$$\text{or, } r^2 + v_t^2 \Delta t^2 = r^2 + r\, a_r \Delta t^2 + \frac{a_r^2 \Delta t^4}{4}$$

Subtracting r^2 from both sides,

$$v_t^2 \Delta t^2 = r\, a_r \Delta t^2 + \frac{a_r^2 \Delta t^4}{4}$$

Dividing both sides by Δt^2

$$v_t^2 = r\, a_r + \frac{a_r^2 \Delta t^2}{4}$$

To find the instantaneous values take the limit as $\Delta t \to 0$.

$$v_t^2 = r\, a_r + \frac{a_r^2\,(0)}{4} = r\, a_r$$

$$\therefore\ a_r = \frac{v_t^2}{r} \quad \text{As was to be demonstrated.}$$

This text does not attempt to teach the processes of differentiating and integrating, but its purpose is to give the student some understanding of how the calculus is used in the study of space.

The definition of the derivative of y with respect to x is, in symbol form, $\frac{dy}{dx} = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$. Any calculus book has a table of derivatives, and there is also one in *The Engineer's Manual* by Ralph G. Hudson on pages 31 and 32.

The average velocity over a period of time, as given in the previous section, is:

$$v = \frac{s_f - s_o}{t_f - t_o}$$

Usually the average velocity is not of direct value in analysis, but the instantaneous velocity is. The speedometer in a car measures instantaneous speed, and if a motorist is arrested for speeding, it is because of his instantaneous velocity, not his average velocity. If s is the path of a particle, its instantaneous velocity is equal to: $\frac{ds}{dt} = \lim\limits_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$.

Example: A particle moves so that its distance from the origin at any time follows the formula $s = t^3$. Find its average and final, velocity and acceleration after 3 seconds.

$$v = \frac{s_f - s_o}{t_f - t_o} \qquad t_o = 0, \quad t_f = 3, \qquad s_o = 0, \quad s_f = 27$$

$$v_{av} = \frac{27 - 0}{3 - 0} = 9 \quad \text{Answer}$$

$$v_f = \frac{ds}{dt} = \frac{d}{dt}(t^3)$$

From page 32 of *The Engineer's Manual:*

$$\frac{d}{dx}(u^n) = n\, u^{n-1} \frac{du}{dx}$$

$$\frac{d}{dt}(t^3) = 3t^{3-1} \frac{dt}{dt} = 3t^2$$

$$v_f = 3(3)^2 = 27 \quad \text{Answer}$$

$$a_{av} = \frac{v_f - v_o}{t_f - t_o} = \frac{27 - 0}{3} = 9 \quad \text{Answer}$$

$$a_f = \frac{dv_f}{dt} = \frac{d(3t^2)}{dt} = 2\,(3t^{2-1})\frac{dt}{dt} = 6t = 18 \quad \text{Answer}$$

Note that with the use of differential calculus, final or instantaneous values for velocity and acceleration can be determined, but only average values can be determined from the formulas given in the previous section.

If, in the example above, the acceleration were given as $a_f = 6t$, the instantaneous velocity and position could be determined by the process of integration. Integral calculus is a summation process that is the inverse of differential calculus.

Example: $a_f = 6t$. Find v_f after 3 seconds. If a curve is drawn with acceleration on the vertical axis and time on the horizontal axis, the area under the curve is the velocity (Fig. 4). Integration gives the sum of all the individual shaded rectangles as

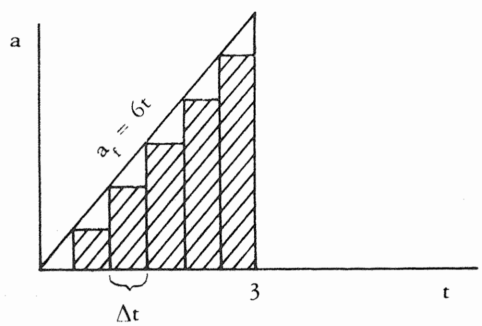


Figure 4. Graph of the continuous function $a_f = 6t$.

Δ t approaches 0 as a limit. As $\Delta t \rightarrow 0$, the area of the rectangles approaches the area under the curve as a limit and is the velocity in this problem. The symbol for integration is $\int$. From the table of fundamental theorems on integrals (Hudson's *The Engineer's Manual,* p. 39),

$$\int u^n du = \frac{u^{n+1}}{n+1} + c$$

In this example problem, the limits of integration, $t = 0$ to $t = 3$, are specified, so the $+ c$ (constant of integration) may be dropped.

$$v_f = \int_0^3 a_f dt = \int_0^3 6t\ dt = \frac{6t^{1+1}}{2} \Big|_0^3$$

$$v_f = 3t^2 \Big|_0^3 = 3(3)^2 - 3(0)^2 = 27 \qquad \text{Answer}$$

The processes of integration and differentiation of variables as applied to the computation of velocity and acceleration through the calculus are part of the study of motion taken up in the branch of dynamics known as kinematics. The study of the forces causing the motion belongs to another branch of dynamics called kinetics.

LAWS OF MOTION

Natural bodies in space follow the basic laws of dynamics, as described by Newton's universal law of gravitation and his three laws of motion. By applying the basic laws and making use of calculus (also developed by Newton), one can explain and prove Kepler's three laws of planetary motion. It would be well to review Kepler's laws before stating Newton's law of universal gravitation, which is one of the laws upon which computation of trajectories and orbits* is based, and Newton's three laws of motion, which describe terrestrial motion as well as celestial mechanics.

Kepler's Laws

From his observations and study, Kepler concluded that the planets travel around the sun in an orbit that is not quite circular. He stated his first law thus: *The orbit of each planet is an ellipse with the sun at one focus.*

Later Newton found that certain refinements had to be made to Kepler's first law to take into account perturbing influences. As the law is applied to manmade satellites, we must assume that perturbing influences like air resistance, the non-spherical (pear shape) shape of the earth, and the influence of other heavenly bodies are negligible. The law as applied to statellites is as follows: *The orbit of a satellite is an ellipse with the center of the earth at one focus.* The path of a ballistic missile, not including the powered and reentry portions, is also an ellipse, but one that happens to intersect the surface of the earth.

Kepler's second law, or law of areas, states: *Every planet revolves so that the line joining it to the center of the sun sweeps over equal areas in equal times.*

To fit earth orbital systems, the law should be restated thus: *Every satellite orbits so that the line joining it with the center of the earth sweeps over equal areas in equal time intervals.*

When the orbit is circular, the application of Kepler's second law is clear, as shown in Figure 5. In making one complete revolution in a circular orbit, a satellite at a constant distance from the center of the earth (radius r) would, for example, sweep out eight equal areas in the total time period ($P = 1$). Each of these eight areas is equal and symmetrical. According to Kepler's second law, the time required to sweep out each of the eight areas is the same. When a satellite is traversing a circular orbit, therefore, its speed is constant.

When the orbit is elliptical rather than circular, the application of Kepler's second law is not quite so easy to see; although the areas are equal, they are not symmetrical (Fig. 6). Note, for example, that the arc of Sector I is much longer than the arc of Sector V. Therefore, since the radius vector sweeps equal

* The terms "trajectory" and "orbit" are sometimes used interchangeably. Use of the term "trajectory" came to astronautics from ballistics, the science of the motion of projectiles shot from artillery or firearms, or of bombs dropped from aircraft. The term "orbit" is used in referring to natural bodies, spacecraft, and manmade satellites. It is the path made by a body in its revolution about another body, as by a planet about the sun or by an artificial satellite about the earth.

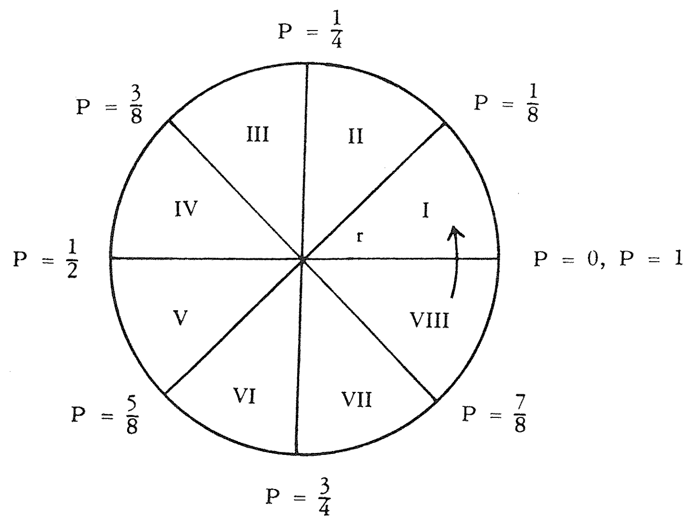


Figure 5. Law of areas as applied to a circular orbit.

areas in equal fractions of the total time period, the satellite must travel much faster around Sector I (near perigee) than around Sector V (near apogee). The perigee (a word derived from the Greek prefix *peri-*, meaning "near," and the Greek root *ge*, meaning "pertaining to the earth") is the point of the orbit nearest the earth. The apogee is that point in the orbit at the greatest distance from the earth (the Greek prefix *apo-* means "from" or "away from").

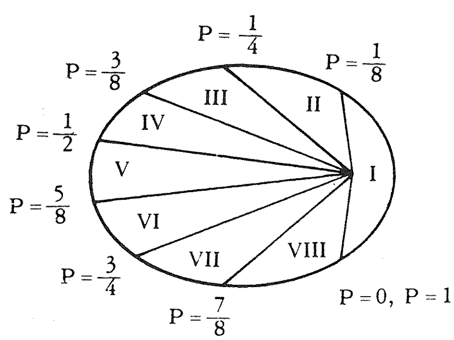


Figure 6. Law of areas as applied to an elliptical orbit.

Kepler's third law, also known as the harmonic law, states: *The squares of the sidereal periods* of any two planets are to each other as the cubes of their mean distances from the center of the sun.*

To fit an earth orbital system, Kepler's third law should be restated thus: *The squares of the periods of the orbits of two satellites are proportional to each other as the cubes of their mean distances from the center of the earth.* The

* The period of a planet about the sun.

mean distance is the length of the semimajor axis (a) of the ellipse, which is an average of the distances to perigee and apogee. Of course, in a circular orbit, the mean distance is the radius, r.

Newton's Laws

While Kepler was working out his three laws of planetary motion, Galileo, an Italian physicist and astronomer, was studying the effects of gravity on falling bodies. Newton drew upon the work of both Kepler and Galileo to formulate his laws of motion.

Newton's first law states: *Every body continues in a state of rest or of uniform motion in a straight line, unless it is compelled to change that state by a force imposed upon it.* In other words, a body at rest tends to remain at rest, and a body in motion tends to remain in motion unless it is acted upon by an outside force. This law is sometimes referred to as the law of inertia.

The second law of motion as stated by Newton says: *When a force is applied to a body, the time rate of change of momentum is proportional to, and in the direction of, the applied force.* If the mass remains constant, this law can be written as $F = Ma$.

Newton's third law of motion is the law of action and reaction: *For every action there is a reaction that is equal in magnitude but opposite in direction to the action.* If body A exerts a force on body B, then body B exerts an equal force in the opposite direction on body A.

Force as Measured in the English System

Newton's three laws of motion are stated in terms of four quantities: force, mass, length, and time. Three of these, length, time and either force or mass, may be completely independent, and the fourth is defined in terms of the other three by Newton's Second Law. Since the units and relative values of these quantities were not known, Newton stated his second law as a proportionality. Assuming that mass does not change with time, this proportionality is stated as $F \propto ma$. If proper units are selected, this statement may be written as an equation:

$$F = ma$$

The following are used in the *metric* system of measurement:

F (dynes) = m (grams) times a (centimeters per second per second)

F (Newtons) = m (kilograms) times a (meters per second per second)

The most common force experienced is that of weight, the measure of the body's gravitational attraction to the earth or other spatial body. Since this attraction is toward the center of the earth, weight, like any force, is a vector quantity. When the only force concerned is weight, the resulting acceleration is normally called "g," the acceleration due to gravity. For this special case, Newton's Second Law can then be written:

$$W = Mg$$

This equation can be used as a definition of mass. The value of g near the surface of the earth is approximately 32.2 feet per second per second; "g" is a vector quantity since it is directed always toward the center of the earth. If the weight, W, is expressed in pounds, rearranging gives:

$$M = \frac{W\ (\text{pounds})}{g\ (\text{feet/sec}^2)}$$

The unit of mass in this equation is called a "slug." Note that mass is a scalar quantity* and is an inherent property of the amount of matter in a body. Mass is independent of the gravitational field, whereas weight is dependent upon the field, the position in the field, and the mass of the body being weighed.

Finally, Newton's Second Law may now be written:

$$F\ (\text{pounds}) = M\ (\text{slugs})\ \text{times}\ a\ (\text{ft/sec}^2)$$

The following example shows the use of this system of units and the magnitude of the "slug":

A package on earth weighs 161 pounds.

Find: (a) its mass in slugs.

(b) the force necessary to just lift it vertically from a surface.

(c) the force necessary to accelerate it 10 ft/sec² on a smooth, level surface.

(d) its weight if it were on the moon; assume the value of "g" there is 1/6 of that value here on the earth.

(e) its mass on the moon.

Solution:

(a) $M\ (\text{slugs}) = \frac{W}{g} = \frac{161\ \text{pounds}}{32.2\ \text{ft/sec}^2} = 5\ \text{slugs}$

(b) $F = W = M\,g = (5\ \text{slugs})\ (32.2\ \text{ft/sec}^2) = 161\ \text{pounds}$
The force must be applied upwards, in the direction opposite to weight.

(c) $F = M\,a = (5\ \text{slugs})\ (10\ \text{ft/sec}^2) = 50\ \text{pounds}$

(d) $W_{moon} = M\,g_{moon} = (5\ \text{slugs})\ (\frac{32.2}{6}\ \text{ft/sec}^2) = 26.83\ \text{pounds}$

(e) $M_{moon} = \frac{W_{moon}}{g_{moon}} = \frac{26.83\ \text{pounds}}{\frac{32.2}{6}\ \text{ft/sec}^2} = 5\ \text{slugs}$

* A scalar quantity has magnitude *only*, in contrast to a vector quantity which has magnitude and direction.

This last solution is, of course, to reemphasize that mass is independent of position. It will be shown later that the local value of g varies with altitude above the earth. It is significant to note that the weight will vary such that the ratio $\frac{W}{g}$ remains constant.

Energy and Work

Work, w, is defined as the product of the component of force in the direction of motion and the distance moved. Thus, if a force, F, is applied and a body moves a distance, s, in the direction the force is applied, $w = Fs$ (Fig. 7). The units of work are foot-pounds. Work is a scalar as distinguished from a vector quantity.

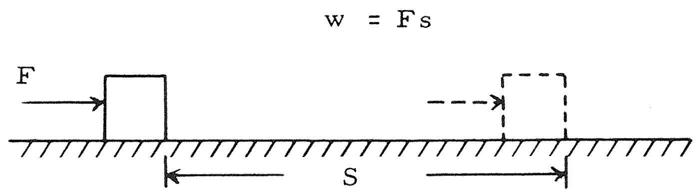


Figure 7. Work performed as a force (F) is moved over the distance s.

To do work against gravity, a force must be applied to overcome the weight, which is the force caused by gravitational acceleration, g.
Therefore, $F = Mg$. If the body is lifted a height h (Fig. 8) and friction is negligible, $w = Mgh$. For problems in which h is mich less than the radius from the center of the earth ($h << r$), g may be considered a constant.

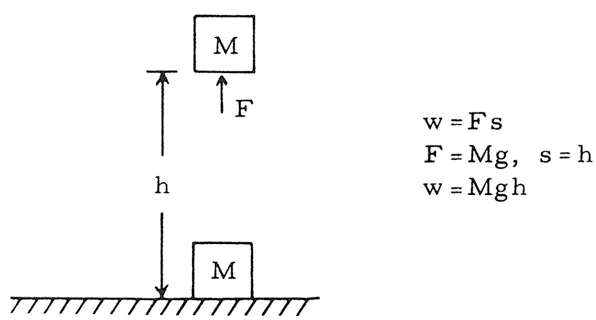


Figure 8. Work performed in lifting.

If an object is pushed up a frictionless inclined plane, the work done is still Mgh (Fig. 9).

$$\begin{aligned} w &= Fs \\ F &= Mg \sin \theta \\ s &= \frac{h}{\sin \theta} \\ w &= Mgh \end{aligned}$$

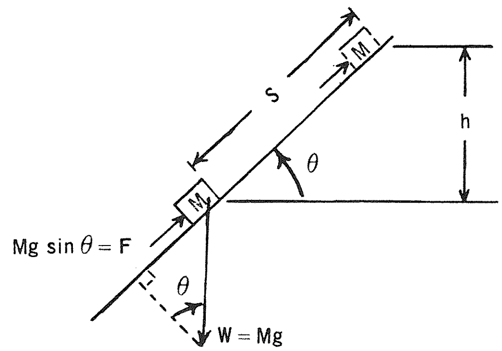


Figure 9. Work performed on a frictionless inclined plane.

For orbital mechanics problems, g varies and must be replaced by the value $\sqrt{g_1 g_2}$ where the subscripts indicate the beginning and final values of g. In such cases

$$w = M\sqrt{g_1 g_2}\, h$$

Another type of work is that work done against inertia. If, in moving from one point to another, the velocity of a body is changed, work is done. This work against inertia is computed in the following steps:

$$\begin{aligned} w &= Fs \\ \text{but, } F &= M a \\ \text{and } 2as &= v_f^2 - v_o^2 \\ s &= \frac{v_f^2 - v_o^2}{2a} \\ \text{so, } w &= Fs = M \frac{a\,(v_f^2 - v_o^2)}{2a} = \frac{M\,(v_f^2 - v_o^2)}{2} = \frac{Mv_f^2}{2} - \frac{Mv_o^2}{2} \end{aligned}$$

The quantity $\frac{Mv^2}{2}$ is defined as kinetic energy (KE). Therefore, work done against inertia (if the altitude and the mass remain the same) is equal to the change in kinetic energy. Energy is defined as the ability to do work, and it is obvious that a moving body has the ability to do work (for example, a moving hammer's ability to drive a nail). A body is also able to do work because of its position or altitude; this is known as potential energy (PE). Units used to measure energy are similar to those used to measure work in that both are scalar rather than vector quantities.

The sum of the kinetic and the potential energy of a body is its *total mechanical energy.*

Newton's Law of Universal Gravitation

Newton published his *Principia* in 1687 and included in it the law of universal gravitation, which he had been considering for about twenty years. This law was based on observations made by Newton. Later work showed that it was only an approximation, but an extremely good approximation. The law states: *Every particle in the universe attracts every other particle with a force that is proportional to the product of the masses and inversely proportional to the square of the distance between the particles.* A constant of proportionality, G, termed the Universal Gravitational Constant, was introduced, and the law was written in this manner:

$$F = \frac{Gm_1 m_2}{r^2}$$

The value of G, the Universal Gravitational Constant, was first determined by Cavandish in a classical experiment using a torsion balance. The value of G is quite small ($G = 6.6695 \times 10^{-8}$ cgs units). In most problems the mass of one of the bodies is quite large. It is convenient, therefore, to combine G and the large mass, m_1, into a new constant, μ (mu), which is defined as the gravitational parameter. This parameter has different values depending upon the value of the large mass, m_1. If m_1 refers to the earth, the gravitational parameter, μ, will apply to all earth satellite problems. However, if the problem concerns satellites of the sun or other large bodies, μ will have a different value based on the mass of that body.

If we now simplify the law of gravitational attraction by combining G and m_1 and by adjusting the results for the English engineering unit system, we obtain the following:

$$G m_1 = \mu \frac{ft^3}{sec^2}$$

$$F = \frac{\mu}{r^2} m \text{ (Where F is lb force and m is slugs)}$$

If this expression is equated to the expression of Newton's Second Law of Motion, as it applies in a gravitational field, we see that:

$$F = mg = \frac{\mu}{r^2} m$$

and after dividing by the unit mass, m, we obtain:

$$g = \frac{\mu}{r^2}$$

Thus, the value of g varies inversely as the square of the distance from the center of the attracting body.

For problems involving earth satellites, the following two constants are necessary for a proper solution:

$$G m_{earth} = \mu_{earth} = 14.08 \times 10^{15} \frac{ft^3}{sec^2}$$

$$r_e \text{ (radius of earth)} = 20.9 \times 10^6 \text{ ft}$$

The formulas must be used with proper concern for the units involved, and the value given for μ applies only to bodies attracted to the earth.

Before applying Newton's law of universal gravitation to the solution of problems, it would be well to consider the possible paths that a body in unpowered flight must follow through space.

CONIC SECTIONS

The conic sections were studied by the Greek mathematicians, and a body of knowledge has accumulated concerning them. They have assumed new significance in the field of astronautics because *any free-flight trajectory can be*

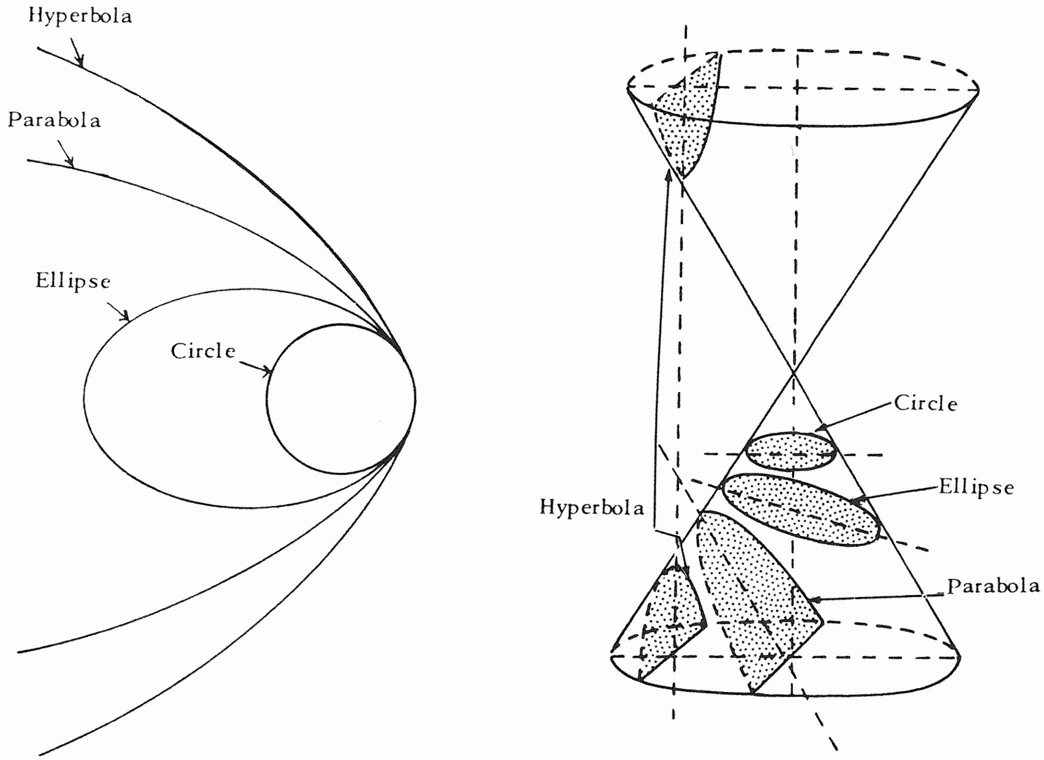


Figure 10. Conic sections.

represented by a conic section. The study of conic sections, or conics, is part of analytic geometry, a branch of mathematics that brings together concepts from algebra, geometry, and trigonometry.

A conic section is a curve formed when a plane cuts through a right circular cone at any point except at the vertex, or center. If the plane cuts both sides of one nappe of the cone, the section is an *ellipse* (Fig. 10). The *circle* is a special case of the ellipse occurring when the plane cuts the cone perpendicularly to the axis. If the plane cuts the cone in such a way that it is parallel to one of the sides of the cone, the section is called a *parabola.* If the plane cuts both nappes of the cone, the section is a *hyperbola* which has two branches.

In one mathematical sense, all conic sections can be defined in terms of eccentricity (ϵ). The numerical value of ϵ is an indication of the relative shape of the conic (rotund or slender) and also an indication of the identity of the conic.

If the eccentricity is zero, the conic is a circle; if the eccentricity is greater than zero but less than one, the conic is an ellipse; if the eccentricity is equal to one, the conic is a parabola; and if the eccentricity is greater than one, the conic is a hyperbola.

Conic Sections and the Coordinate Systems

In locating orbits or trajectories in space, it is possible to make use of either rectangular (sometimes called Cartesian) or polar coordinates. In dealing with artificial satellites, it is often more convenient to use polar rather than rectangular coordinates because the center of the earth can be used both as the origin of the coordinates and as one of the foci of the ellipse.

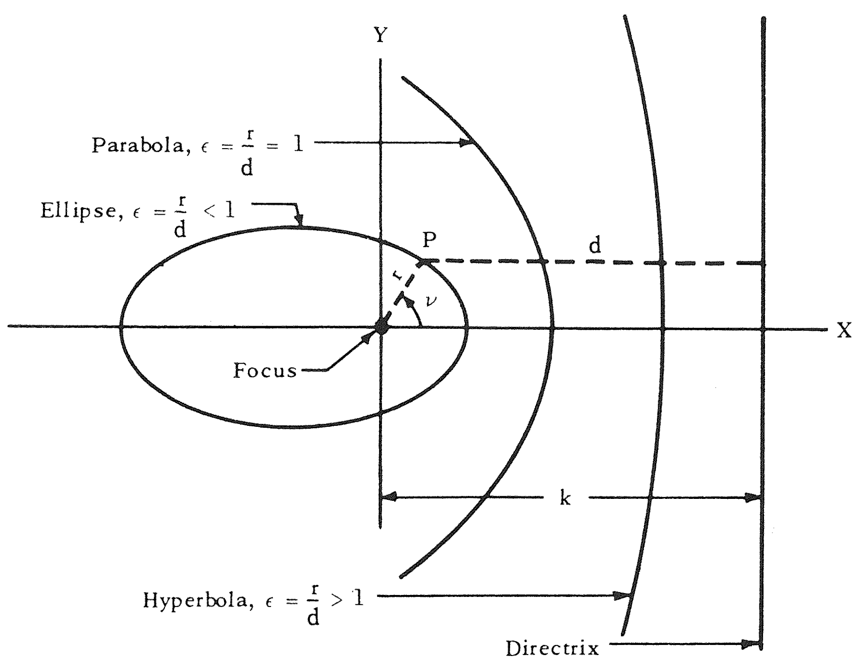


Figure 11. Rectangular and polar coordinates superimposed on the conic sections.

If rectangular and polar coordinates are superimposed upon a set of conics as shown in Figure 11, equations of the curves can be derived.

The formula for the eccentricity of a conic is $\epsilon = \frac{r}{d}$. This ratio is constant for a specific curve.

In Cartesian coordinates:

$$\epsilon = \frac{r}{d} = \frac{\sqrt{x^2 + y^2}}{k - x}$$

$$\sqrt{x^2 + y^2} = \epsilon(k - x)$$

Squaring both sides gives:

$$x^2 + y^2 = \epsilon^2(k - x)^2$$

To convert the rectangular coordinates to polar coordinates, substitute as follows:

$$x = r \cos \nu \text{ (ν is the lower case Greek nu)}$$

$$y = r \sin \nu$$

$$\epsilon = \frac{r}{d} = \frac{r}{k - r \cos \nu}$$

$$k\epsilon - r\epsilon \cos \nu = r$$

$$r + r\epsilon \cos \nu = k\epsilon$$

$$r = \frac{k\epsilon}{1 + \epsilon \cos \nu}$$

This result is the general equation for *all* conics.

Ellipse

The ellipse is the curve traced by a point (P) moving in a plane such that the sum of its distances from two fixed points (foci) is constant. In the ellipse in Figure 12, the following are shown: the foci (F and F′); c, distance from origin to either focus; a, distance from origin to either vertex (semimajor axis); 2a, major axis; b, distance from origin to intercept on y-axis (semiminor axis); 2b, minor axis; and r + r′, distances from any point (P) on the ellipse to the respective foci (F and F′).

A number of relationships which are very useful in astronautics are derived from the geometry of the ellipse:

$$r + r' = 2a \text{ (at any point on the ellipse)}$$

$$a^2 = b^2 + c^2 \text{ or } a = \sqrt{b^2 + c^2}$$

$$b = \sqrt{a^2 - c^2}$$

$$c = \sqrt{a^2 - b^2}$$

The eccentricity of the ellipse $(\epsilon) = \frac{c}{a}$. A chord through either focus perpendicular to the major axis is called the *latus rectum* and its length $= \frac{2b^2}{a}$.

These relationships can be used to determine the parameters of an elliptical orbit of a satellite when only the radius of perigee and the radius of apogee are known. These parameters are important because, as is shown later, they

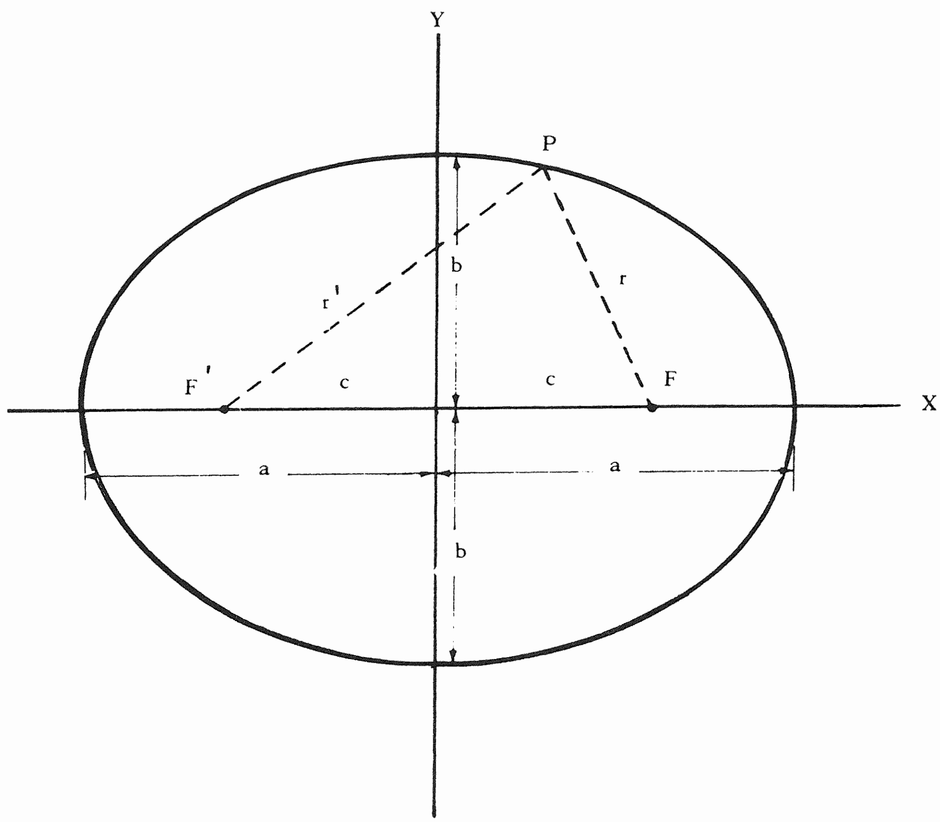


Figure 12. Ellipse with center at origin of rectangular coordinate system.

are related to the total mechanical energy and total angular momentum of the satellite. Thereby they offer a means of determining these values through the simple arithmetic of an ellipse rather than the vector calculus of celestial mechanics.

Sample problem: A satellite in a transfer orbit has a perigee at 300 NM above the surface of the earth and an apogee at 19,360 NM. Find a, b, c, and ϵ for the ellipse traced out by this satellite.

Solution:

Since the center of the earth is one focus of the ellipse, first convert the apogee and perigee to radii by adding the radius of the earth (3440 NM):

$$\begin{aligned} \text{radius of perigee } r_p &= \text{altitude of perigee} + \text{radius of earth} \\ &= 300 + 3440 = 3740 \text{ NM.} \\ \text{radius of apogee } r_a &= \text{altitude of apogee} + \text{radius of earth} \\ &= 19{,}360 + 3440 = 22{,}800 \text{ NM.} \end{aligned}$$

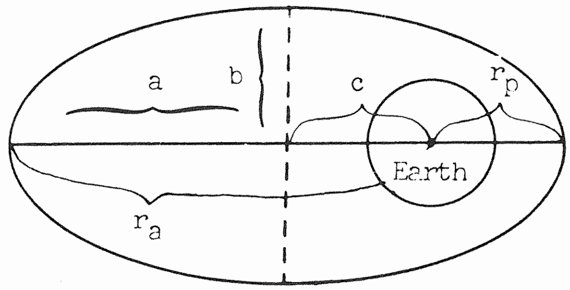


Figure 13. Orbit of an artificial satellite showing radius of perigee and radius of apogee (not to scale).

With this information, an exaggerated sketch of the ellipse can be made (Fig. 13). Compare this with Figure 12 to obtain:

$$r_p + r_a = \text{major axis} = 2a$$

then $\quad 2a = 3740 + 22{,}800 = 26{,}540$ NM

or $\quad a = \dfrac{26{,}540}{2} = 13{,}270$ NM

Also from comparing Figures 12 and 13:

$$\begin{aligned} c &= a - r_p \\ &= 13{,}270 - 3740 = 9530 \text{ NM} \end{aligned}$$

Since a and c are known, find b from the relationship given:

$$\begin{aligned} b &= \sqrt{a^2 - c^2} \\ \text{or } b &= \sqrt{(1.327 \times 10^4)^2 - (.953 \times 10^4)^2} \\ b &= \sqrt{(1.761 \times 10^8) - (.908 \times 10^8)} = \sqrt{.853 \times 10^8} \\ b &= .923 \times 10^4 = 9230 \text{ NM} \end{aligned}$$

According to the formula given for eccentricity:

$$\epsilon = \frac{c}{a}$$

$$\epsilon = \frac{9530}{13270} = .718$$

The ellipse is a conic section with eccentricity less than 1 ($\epsilon < 1$).

CIRCLE. The circle is a special case of an ellipse in which the foci have merged at the center; thus $\epsilon = 0$. The ellipse relationships can be used for a circle.

ENERGY AND MOMENTUM

Once the basic geometry of a trajectory or orbit is understood, the next subject for investigation is the physics of energy and momentum. From concepts of linear and angular motion, concepts of linear and angular momenta logically follow. Once the formulas for computing the specific angular momentum and the specific mechanical energy of a body in orbit are delineated, then it is possible to solve for unknown quantities, such as the altitude of the body above the surface of the earth or the velocity at any point on the orbit. Any body in space following a free-flight path—whether it is a missile, a satellite, or a natural body—is governed by the laws of the conservation of specific mechanical energy and specific angular momentum. Once the value of either of these items is known at any point along a free-flight trajectory or orbit, then its value is known at all other points, since the value does not change unless the body is acted upon by some outside force.

Mechanical Energy

The law of Conservation of Energy states that *energy can neither be created nor destroyed but only converted from one form to another.* This law can be applied to orbital mechanics and restated in this way: *The total mechanical energy of an object in free motion is constant, provided that no external work is done on or by the system.* During reentry, work is done by the system and some of the mechanical energy is converted to heat. Similarly, during launch, work is done on the system as the propulsion units give up chemical energy. In this chapter, only the free-flight portion of the trajectory is considered, and it is assumed that there is no thrust and no drag.

In order to establish a common understanding about changes in the amount of energy, it is necessary to agree upon a zero reference point for energy. Potential energy, or energy due to position, can be, and often is, measured from sea level. In working with earth-orbiting systems, however, the convention is to consider a body as having zero potential energy if it is at an infinite distance from the earth and as having zero kinetic energy if it is absolutely at rest with respect to the center of the earth. Under these circumstances, the total mechanical energy (PE + KE) is also equal to zero. If the total mechanical energy is positive—that is, larger than zero—the body has enough energy to escape from the earth. If the total mechanical energy is negative—that is, less than zero—the body does not have enough energy to escape from the earth, and it must be either in orbit or on a ballistic trajectory.

The formula for PE, with the reference system as stated above, is $PE = \frac{-m\mu}{r}$. Instead of using PE, a specific PE (PE per unit mass) can be used if both sides are divided by m; for example

$$\frac{PE}{m} = \frac{-\mu}{r}$$

If a body is at infinity, it has a specific PE equal to $-\frac{\mu}{r} = \frac{\mu}{\infty} = 0.$

A similar case can be presented for kinetic energy. A body with some velocity relative to the center of the earth has kinetic energy defined by:

$$KE = \frac{mv^2}{2}$$

Again, the specific kinetic energy (kinetic energy per unit mass) can be defined as:

$$\text{Specific KE} = \frac{KE}{m} = \frac{v^2}{2}$$

In general, a body in free motion in space has a particular amount of mechanical energy, and this amount is constant because of the conservation of mechanical energy.

$$\text{Total Mechanical Energy} = KE + PE$$

A more useful expression is obtained if we define Specific Mechanical Energy, E, or the Total Mechanical Energy per unit mass. Thus, we can write:

$$E = \frac{\text{Total Mechanical Energy}}{m}$$

$$E = \frac{KE}{m} + \frac{PE}{m}$$

$$E = \frac{v^2}{2} - \frac{\mu}{r}$$

Specific Mechanical Energy, E, is also conserved in unpowered flight in space. The units of E are $\frac{ft^2}{sec^2}$. Since the mass term does not appear directly in the equation, E represents the specific mechanical energy of a body in general.

If the solution to the Specific Mechanical Energy equation yields a negative value for E, the body is on an elliptical or circular path (nonescape path). If E is exactly equal to zero, the path is parabolic; this is the minimum energy escape path. If E is positive, the path is hyperbolic, and the body will also escape from the earth's gravitational field.

Although the value of E, once determined, remains constant in free flight, there is a continuous change in the values of specific kinetic energy and specific potential energy. High velocities nearer the surface of the earth, representing high specific kinetic energies, are exchanged for greater specific potential energies as distance from the center of the earth increases. In general, velocity is traded for altitude; kinetic energy is traded for potential energy. The sum remains constant.

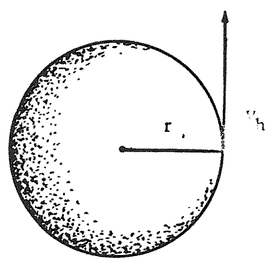
Linear and Angular Momentum

When a body is in motion, it has momentum. Momentum is the property a body possesses because of its mass and its velocity. In linear motion, momentum is expressed as mv and has the units, $\frac{\text{foot-slug}}{\text{sec}}$.

When a rigid body, such as a flywheel, rotates about a center, it has angular momentum. Once the flywheel is in motion, its angular momentum would remain constant if it were not acted upon by forces such as friction and air resistance. Similarly, a gyroscope would rotate indefinitely in the absence of friction and air resistance. Thus, ignoring such losses, angular momentum will remain constant. In space, it can be assumed that such forces are negligible and that angular momentum is conserved. This is another tool to use in analyzing orbital systems.

Angular momentum is the product of moment of inertia, I, and the angular velocity, ω. Moment of inertia of a body of mass, m, rotating about a center at a distance, r, can be expressed as $m\,r^2$. The angular momentum is then equal to $m\,r^2\,\omega$.

For convenience in calculations, the term Specific Angular Momentum, H, is defined as the angular momentum per unit mass. Remembering that the magni-



$$\text{Angular Momentum} = m\,r^2\,\omega$$

$$\omega = \frac{v_{horizontal}}{r}$$

$$H = \frac{\text{Ang. Momentum}}{m}$$

$$\therefore H = v_h r \text{ (Circular Motion)}$$

Figure 14. Specific angular momentum of a circular orbit.

tude of the instantaneous velocity vector of a body rotating in constant circular motion about a center with radius r is equal to ωr and that the vector is perpendicular to the radius, the expression for specific angular momentum of a circular orbit can be simplified as shown in Figure 14.

The general application of specific angular momentum to all orbits requires that the component of velocity perpendicular to the radius vector be used. This velocity component is defined as

$$v_h = v \cos \phi$$

where ϕ is the angle the velocity vector makes with the local horizontal, a line perpendicular to the radius. In an elliptical orbit, the geometry is as shown in Figure 15. The body in orbit has a total velocity v which is always tangent to the flight path.

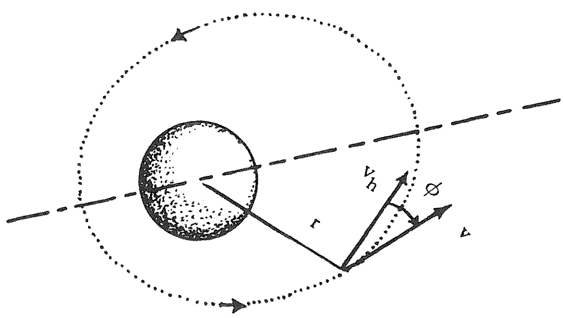
The formula, $H = v\,r\cos\phi$, defines the specific angular momentum for all orbital cases. The angle ϕ *is the flight path angle* and is the angle between the local horizontal and the total velocity vector. It should be noted that the angle ϕ is equal to zero for circular orbits. Further, in elliptical orbits, ϕ is zero at the points of apogee and perigee.

The two important formulas that have been presented in this section are those for E and H. These formulas permit a trajectory or an orbit to be completely defined from certain basic data:

$$E = \frac{v^2}{2} - \frac{\mu}{r}, \text{ when units of E are } \frac{ft^2}{sec^2}$$

$$H = v\,r\cos\phi, \text{ when units of H are } \frac{ft^2}{sec}$$

If v, r, and ϕ are known for a given trajectory (or orbit) at a given position, then E and H can be determined. In the absence of outside forces, E and H are constants; therefore v, r, and ϕ can be determined at any other position on the trajectory or orbit. Equations for the specific angular momentum and the specific mechanical energy can be used in practical application to the two-body problem and to the free-flight portion of the ballistic missile trajectory.



$H = v_h r$

but $v_h = v\cos\phi$

$\therefore H = v\,r\cos\phi$ (General orbit)

Figure 15. Specific angular momentum.

THE TWO-BODY PROBLEM

It is implicit in Newton's law of universal gravitation that every mass unit in the universe attracts, and is attracted by, every other mass unit in the universe. Clearly, small masses at large distances are infinitesimally attracted to each other. It is neither feasible nor necessary to consider mutual attractions of a large number of bodies in many astronautics problems. The most frequent problems of astronautics involve only two interacting bodies: a missile payload, or satellite, and the earth. In these instances, the sun and moon effects are negligible except in the case of a space probe, which will be noticeably affected by the moon, if it passes close to the moon, and which will be controlled by the sun, if it escapes from the earth's gravitational field.

Military officers concerned with operational matters are primarily interested in launching a missile from one point on the earth's surface to strike another point on the earth's surface and in launching earth satellites. In these problems, the path followed by the payload is adequately described by considering only two bodies, the earth and the payload. The problem of two bodies is termed the two-body problem; its solution dates back to Newton.

It is indeed fortunate that the solution of the two-body trajectory is simple and straightforward. A general solution to a trajectory involving more than two bodies does not exist. Special solutions for these more complex trajectories usually require machine calculation.

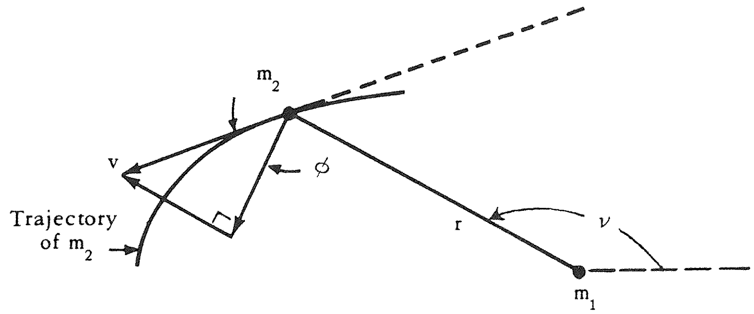


Figure 16. Trajectory relationships.

The two-body problem is described graphically in Figure 16. A small body, m_2, has a velocity, v, at a distance r from the origin chosen as the center of mass of a very massive body, m_1. The problem is to establish the path followed by body, m_2, or to define its trajectory. This is a typical problem in mechanics—given the present conditions of a body, what will these conditions be at any time, t, later? First, we shall find r as a function of ν, where ν is the polar angle measured from a reference axis to the radius vector.

At the outset, it should be apparent that the entire trajectory will take place in the plane defined by the velocity vector and the point origin. There are no forces causing the body, m_2, to move out of this plane; otherwise, the conditions are not those of a two-body free-flight problem.

In the earlier outline of the laws of conservation of energy and momentum, the following conditions were established:

$$\frac{v^2}{2} - \frac{\mu}{r} = E = \text{a constant} \tag{1}$$

$$H = vr \cos \phi = \text{a constant} \tag{2}$$

Equations (1) and (2) can be combined and, with the aid of the calculus, the following equation can be derived:

$$r = \frac{H^2/\mu}{1 + \sqrt{1 + \frac{2EH^2}{\mu^2}} \cos \nu} \tag{3}$$

Equation (3) is the equation of a two-body trajectory in polar coordinates.

Earlier, the following equation was given as the equation of any conic section in polar coordinates, the origin located at a focus:

$$r = \frac{k\epsilon}{1 + \epsilon \cos \nu} \tag{4}$$

Equations (3) and (4) are of the same form; hence, equation (3) is also the equation of any conic section (origin at a focus) in terms of the physical constants, E and H, and the two-body trajectories are then conic sections. This conclusion substantiates Kepler's first law. In fact, Kepler's first law is a special case because an ellipse is just one form of conic section.

Since equations (3) and (4) are of the same form, it is possible to equate like terms, which will lead to relationships between the physical constants, E, H, and μ, and the geometrical constants, ϵ, a, b, and c. Thus:

$$k\epsilon = \frac{H^2}{\mu} \tag{5}$$

and

$$\epsilon = \sqrt{1 + \frac{2EH^2}{\mu^2}} \tag{6}$$

Physical Interpretation of the Two-Body Trajectory Equation

Analysis of the two-body trajectory equation will give an understanding of the physical reaction of a vehicle (small body) under the influence of a planet (large body).

If $E < 0$, the trajectory is an ellipse. What is the condition that E be less than zero? It is simply that the kinetic energy of the small mass, m_2, because of its relatively low velocity, is less than the magnitude of its potential energy. Therefore, the body cannot possibly go all the way to infinity; that is, it cannot go to a point where it is no longer attracted by the larger body—where the potential energy is zero. The smaller mass cannot escape. It must remain "captured" by the force field of the larger body. Therefore, it will be turned back toward the larger body, or, more in keeping with the idea of potential energy, it will always be "falling back" toward the more massive body. When this particular balance of energy exists, the trajectory is elliptical with one focus coincident with the center of mass of the larger body. In the actual physical case, the larger body will have a finite size; that is, it will not be a point mass, and this ellipse may intersect the surface of the larger body as it does in the case of a ballistic missile. If the velocity is sufficiently high, and its direction proper, the ellipse may completely encircle the central body, the condition of a satellite.

If $E = 0$, the kinetic energy exactly equals the magnitude of the potential energy, and the small mass, m_2, has just enough energy to travel to infinity, away from the influence of the central body, and come to rest there. The small body will follow a parabolic path to infinity. The velocity which is associated with this very special energy level is also very special and is commonly called the "escape velocity."

Escape velocity can be calculated by setting $E = 0$ in the mechanical energy equation (1) as follows:

$$\frac{V^2_{esc}}{2} - \frac{\mu}{r} = E = 0$$

$$V^2_{esc} = 2\left(\frac{\mu}{r}\right)$$

$$V_{esc} = \sqrt{\frac{2\mu}{r}} \tag{7}$$

Thus, it can be seen that escape velocity decreases with distance from the center of the earth. At the earth's surface,

$$v_{esc} = \sqrt{\frac{2\mu}{r_e}} = \left[\frac{(2)\,(14.08)\,\left(10^{15}\,\frac{ft^3}{sec^2}\right)}{(20.9)\,(10^6\,ft)}\right]^{\frac{1}{2}}$$

$$v_{esc} = 36{,}700 \text{ ft/sec.}$$

If the velocity of the small mass exceeds escape velocity, which will be the case if $E > O$, it will follow a hyperbolic trajectory to infinity. In practice infinity is a large distance at which the earth's attractive force is insignificant, and there the mass will have some residual velocity. In a mathematical sense, the body would still have velocity at infinity. In a physical sense, it would have velocity relative to the earth at any large distance from the earth.

Considering the sounding rocket, only the straight-line, degenerate conic is a possible trajectory. But, again, the value of E will determine whether escape is possible; that is, if $E < O$, the straight-line trajectory cannot extend to infinity. If $E = O$ or $E > O$, the straight line will extend to infinity.

Example Problem: The first U.S. "moon shot," the Pioneer I, attained a height of approximately 61,410 NM above the earth's surface. Assuming that the Pioneer had been a sounding rocket (a rocket fired vertically), and assuming a spherical, nonrotating earth without atmosphere, calculate the following:

a. E (total specific energy)

b. Impact velocity (earth's surface)

Solution: Given

(a) At apogee (greatest distance from earth):

Altitude (above earth's surface) = 61,410 NM

Earth radius = 3440 NM

Velocity = O (Only for a sounding rocket)

r = altitude + earth's radius

$r = (61{,}410 + 3440)\text{ NM} = 64{,}850\text{ NM}$

$$E = \frac{v^2}{2} - \frac{\mu}{r} = 0 - \frac{(14.08)\,(10^{15})\,\frac{ft^3}{sec^2}}{(64{,}850)\text{ NM }(6080)\,\frac{ft}{NM}}$$

$$E = -\,3.57 \times 10^7 \text{ ft}^2/\text{sec}^2$$

Answer

(b) Since the specific energy is constant,

At the earth's surface:

$r = 3440$ NM

$E = -\,3.57 \times 10^7 \text{ ft}^2/\text{sec}^2$

$$E = \frac{v^2}{2} - \frac{\mu}{r}$$

$$-3.57 \times 10^7 \frac{ft^2}{sec^2} = \frac{v^2}{2} - \frac{(14.08)\,(10^{15}\frac{ft^3}{sec^2})}{(3440)\ NM\ (6080)\ \frac{ft}{NM}}$$

$$v^2_{\,(impact)} = 2\,[(-3.57 \times 10^7) + (67.4 \times 10^7)] = 2(63.8 \times 10^7) ft^2/sec^2$$

$$v_{\,(impact)} = 35{,}700\ ft/sec \qquad \text{Answer}$$

This is also the approximate burnout velocity of the vehicle. As the surface escape velocity is 36,700 ft/sec, it is clear that Pioneer I did not attain escape velocity, and so it returned to earth.

Elliptical Trajectory Parameters

While parabolic and hyperbolic trajectories, especially the latter, are of interest in problems of interplanetary travel, elliptical trajectories comprise the ballistic missile and satellite cases, which are of current military interest. It is important, then, to relate the dimensions of an ellipse (a, b, and c) to the physical constants (E, H, and μ) as was previously done for ϵ.

The relationship, $r_a + r_p = 2a$, was presented earlier. If this equation is applied to point P in Figure 17,

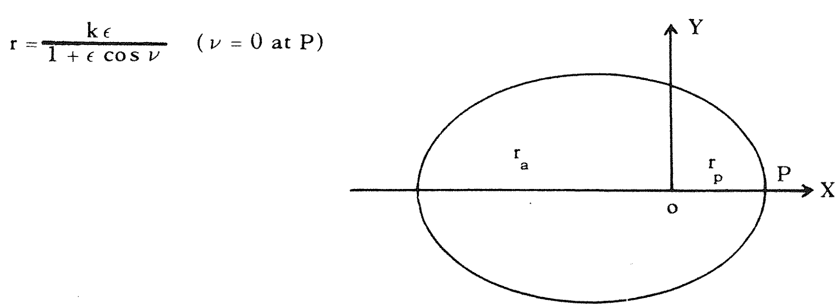


Figure 17. Ellipse.

$$r = \frac{k\epsilon}{1 + \epsilon} \text{ and } r_p = a - c. \text{ Then,}$$

$$\frac{k\epsilon}{1 + \epsilon} = a - c \tag{8}$$

But $\epsilon = \frac{c}{a}$; therefore, substituting $c = \epsilon a$ into (8),

$$\frac{k\epsilon}{1 + \epsilon} = a - \epsilon a = a(1 - \epsilon)$$

$$k\epsilon = a(1 - \epsilon^2) \tag{9}$$

But from equations (5) and (6),

$$k\epsilon = H^2/\mu \text{ and } \epsilon = \sqrt{1 + \frac{2EH^2}{\mu^2}}$$

Substituting these relationships into equation (9),

$$\frac{H^2}{\mu} = a\left(1 - 1 - \frac{2EH^2}{\mu^2}\right)$$

$$\frac{H^2}{\mu} = -\frac{2EH^2 a}{\mu^2}$$

$$-1 = \frac{2Ea}{\mu}$$

$$a = -\frac{\mu}{2E} \quad \text{(EXTREMELY USEFUL)} \tag{10}$$

$$\text{Also } a = -\frac{\mu}{2\left(\frac{v^2}{2} - \frac{\mu}{r}\right)} \tag{11}$$

From the following equation:

$$\frac{b^2}{a^2} = 1 - \epsilon^2$$

$$\frac{b^2}{a} = a(1 - \epsilon^2)$$

But from equation (9),

$$a(1 - \epsilon^2) = k\epsilon = \frac{H^2}{\mu}$$

Therefore,

$$\frac{b^2}{a} = \frac{H^2}{\mu} \tag{12}$$

Equations (10) and (12) are extremely important relationships; an understanding of them is essential to material that follows on ballistic missiles and satellites. If injection conditions of speed and radius are fixed, it is clear that a, the semimajor axis of the elliptical trajectory, becomes fixed, regardless of the value of the flight path angle at burnout. Equation (10) points out there is a direct relationship between the size of an orbit and the energy level of the orbiting object. Equation (12) points out that for a given energy level there is a direct relationship between the length of the semi latus rectum of an elliptical trajectory (a shape parameter) and the specific angular momentum of the orbiting object. This implies that the size and shape of an elliptical trajectory are determined by the E and H.

Two-Body Trajectory Definitions and Geometry

The general equation of two-body trajectories has now been introduced. Before proceeding to problem applications it would be well to consider in detail some commonly used terms and symbols.

First, refer to Figure 18. In general, the point P is called the periapsis and P′ the apoapsis. If the earth is at point O, the ellipse would then represent the trajectory of an earth satellite; P is then termed perigee and P′ apogee. If the sun is at point O, the ellipse would represent a planetary orbit; P is then called perihelion and P′ aphelion.

In order to explain the use of the angle ν, the geometry of satellites will be discussed briefly. Figure 18 depicts a planetary orbit (not to scale).

In astronomy and celestial mechanics it is standard practice to measure a body's position from perihelion point P. There are several reasons for using perihelion, including the fact that perihelion of any body in the solar system except Mercury and Venus is closer to the earth's orbit than is the body's aphelion. In fact, for a highly eccentric orbit such as a comet's, the body would not be visible at aphelion. In order to conform to accepted practice, then, the angle nu (ν), measured from periapsis, has been introduced. This angle, which is called the *true anomaly,* is of considerable importance in time-of-flight calculations.

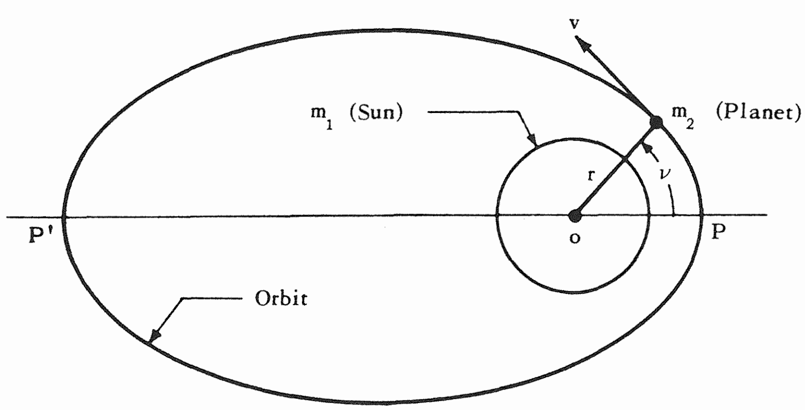


Figure 18. Sun-centered orbit

The *true anomaly,* however, does not lend itself well to ballistic missile problems, as can be seen from the simplified ballistic missile geometry in Figure 19.

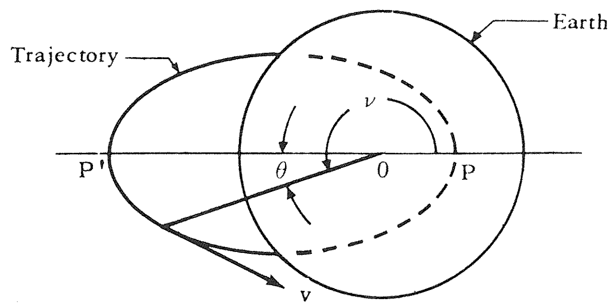


Figure 19. Ballistic missile trajectory.

In a ballistic missile trajectory, perigee is entirely fictitious. The missile obviously never traverses the dashed portion of the trajectory. The solid portion of the trajectory is all that is of real interest, and this portion is in the second and third quadrants of the angle ν. It is convenient then to define an angle θ, measured counterclockwise from apogee (apoapsis, in general), such that

$$\nu = \theta + \pi. \tag{13}$$

θ will normally have values in the first and fourth quadrants. From (13), it is clear that derivatives of ν and θ will be interchangeable. The equation of a conic section in terms of θ can be found by substituting (13) into (4),

$$r = \frac{k\epsilon}{1 + \epsilon \cos \nu} = \frac{k\epsilon}{1 + \epsilon \cos (\theta + \pi)}$$

$$r = \frac{k\epsilon}{1 - \epsilon \cos \theta}.$$

With this understanding of the relationship between ν and θ, it will be convenient to use ν when working with satellite and space trajectories and θ when working with ballistic missiles (see App. C).

EARTH SATELLITES

During their free flight trajectory, satellites and ballistic missiles follow paths described by the two-body equation. For a satellite to achieve orbit, enough energy must be added to the vehicle so that the ellipse does not intersect the surface of the earth. However, not enough energy is added to allow the vehicle to escape. Therefore, the ellipse and the circle are the paths of primary interest.

The orientation, shape, and size of orbits are important to the accomplishment of prescribed missions. Therefore, eccentricity (ϵ), major axis (2a), minor axis (2b), and distance between the foci (2c) are of interest. It is necessary to know the relationships of these geometric values to the orbital parameters in order to make an analysis of an orbit. For example, it is helpful to remember that:

r_p (radius at perigee) $= a - c$

r_a (radius at apogee) $= a + c$

$$r_p + r_a = 2a$$

$$\epsilon = \frac{c}{a}$$

$$a^2 = b^2 + c^2$$

Specific mechanical energy, E, and specific angular momentum, H, are of primary concern when elliptical and circular orbits are discussed. If there are no outside forces acting on a vehicle in an orbit, the specific mechanical energy and the specific angular momentum will have constant values, regardless of position in the orbit. This means that if E and H are known at one point in the orbit, they are then known at each and every other point in the orbit. At a given position if radius r, speed v, and flight path angle ϕ are known, E and H can be determined from:

$$E = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

$$H = vr \cos \phi$$

If the values of E and H are known for a particular orbit, and the speed and flight path angle at a certain point in the orbit are to be determined, the energy equation can be solved for v, and then the angular momentum equation can be solved for ϕ.

The equations for the speed in circular and elliptical orbits are important. The equation for circular speed is:

$$v = \sqrt{\frac{\mu}{r}}$$

The equation for elliptical speed is:

$$v = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}}$$

Another equation that is important in the analysis of orbits is the equation for orbital period. For a circular orbit the distance around is the circumference of the circle which is $2\pi r$. Therefore, the period, which is equal to the distance around divided by the speed, is this:

$$P = \frac{2\pi r}{v}.$$

Now, substitute for v the speed in circular orbit:

$$v = \sqrt{\frac{\mu}{r}}$$

$$P = \frac{2\pi r}{\sqrt{\frac{\mu}{r}}}$$

Multiply numerator and denominator of the right hand side by $\sqrt{r}$:

$$P = \frac{2\pi r \sqrt{r}}{\sqrt{\frac{\mu}{r}}\sqrt{r}} = \frac{2\pi r^{3/2}}{\sqrt{\mu}}$$

Using the principle of Kepler's third law, replace r by the mean distance from the focus, which is equal to the semimajor axis a, and the equation becomes:

$$P = \frac{2\pi a^{3/2}}{\sqrt{\mu}}$$

Squaring both sides, $P^2 = \frac{4\pi^2 a^3}{\mu}$. Since $\frac{4\pi^2}{\mu}$ is a constant, P^2 is proportional to a^3, and for earth satellites $P^2 = \left(2.805 \times 10^{-15} \frac{sec^2}{ft^3}\right)(a)^3$. Or, $P = \left(5.30 \times 10^{-8} \frac{sec}{ft^{\frac{3}{2}}}\right)(a)^{\frac{3}{2}}$.

Problem: Initial data from Friendship 7 indicated that the booster burned out at a perigee altitude of 100 statute miles, speed of 25,700 ft/sec, and flight path angle of 0°. Determine the speed and height at apogee, and the period.

Given*: $h_p = 100 \text{ SM} = .5 \times 10^6 \text{ ft}$ $\qquad r_e = 20.9 \times 10^6 \text{ ft}$

$v_{bo} = 25{,}700 \text{ ft/sec}$

$\phi_{bo} = 0°$

Find: v_a, h_a, P

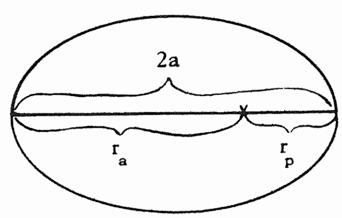


Figure 20. Orbit of Friendship 7 (not to scale).

* Even though $\phi_{bo} = 0°$, if burnout altitude were not given as perigee altitude, you would have to determine if this were perigee or apogee. To do this, you would compute the circular speed for the given burnout altitude and compare this with the actual speed. If the circular speed is greater than the actual, burnout was at apogee; if the circular speed was less than the actual speed, burnout was at perigee.

Solution:

$$E = \frac{v^2}{2} - \frac{\mu}{r} = \frac{(2.57 \times 10^4)^2}{2} - \frac{14.08 \times 10^{15}}{(20.9 + .5) \times 10^6}$$

$$= (3.31 \times 10^8) - (6.58 \times 10^8) = -3.27 \times 10^8$$

$$\text{but } E = -\frac{\mu}{2a};\ 2a = -\frac{\mu}{E} = \frac{-14.08 \times 10^{15}}{-3.27 \times 10^8}$$

$$= 43.1 \times 10^6$$

From Figure 20, $r_a + r_p = 2a$

$$\therefore r_a = 2a - r_p = (43.1 - 21.4) \times 10^6$$

$$= 21.7 \times 10^6$$

$$h_a = r_a - r_e = (21.7 - 20.9) \times 10^6 = .8 \times 10^6 \text{ ft}$$

$$= 151 \text{ sm} \qquad \text{Answer}$$

$$H_p = H_a$$

$$v_p r_p = v_a r_a \qquad v_a = \frac{v_p r_p}{r_a}$$

$$v_a = \frac{(2.57 \times 10^4)(21.4 \times 10^6)}{21.7 \times 10^6} = 25{,}400 \text{ ft/sec}$$

$$= 17{,}300 \text{ mph} \qquad \text{Answer}$$

$$P^2 = \frac{4\pi^2 a^3}{\mu} = (2.805 \times 10^{-15})(21.6 \times 10^6 \text{ ft})^3 \frac{\text{sec}^2}{\text{ft}^3}$$

$$= 28.1 \times 10^6 \text{ sec}^2$$

$$P = 5.30 \times 10^3 \text{ sec} = 88.3 \text{ min Ans.}$$

It is interesting to compare the computed apogee and period results with the actual orbit (later data gave a higher accuracy for burnout conditions):

Item	*Actual figures*	*Computed figures*
V_{bo}	25,728 ft/sec	25,700 ft/sec
h_{bo}	97.695 SM	100 SM
h_a	158.85 SM	151 SM
P	88.483 min	88.3 min

Note that using three significant figures results in $h_p = .5 \times 10^6$ ft, about 94 SM. Such errors are common using slide rule accuracy, but this problem does illustrate the techniques used.

From the foregoing problem, it is evident that the principles and relatively simple algebraic expressions presented thus far are extremely important. They enable one to analyze a trajectory or orbit rather completely—with a slide rule for academic or generalized discussion purposes, or with a digital computer for system design and operation. The discussion has been, however, confined to the two dimensional orbital plane. Before discussing some of the more interesting facets of orbital mechanics, it is necessary to properly locate a payload in three dimensions.

LOCATING BODIES IN SPACE

In one of the coordinate systems for space used by engineers and scientists, the origin is the center of the earth. This is a logical choice since the center of the earth is a focus for all earth orbits.

With the center of the coordinate systems established, a reference frame is required on which angular measurements can be made with respect to the center. The reference frame should be regular in shape, and it should be fixed in space. A sphere satisfies the requirement of a regular shape. The sphere of the earth would be a handy reference if it were fixed in space, but it rotates constantly.

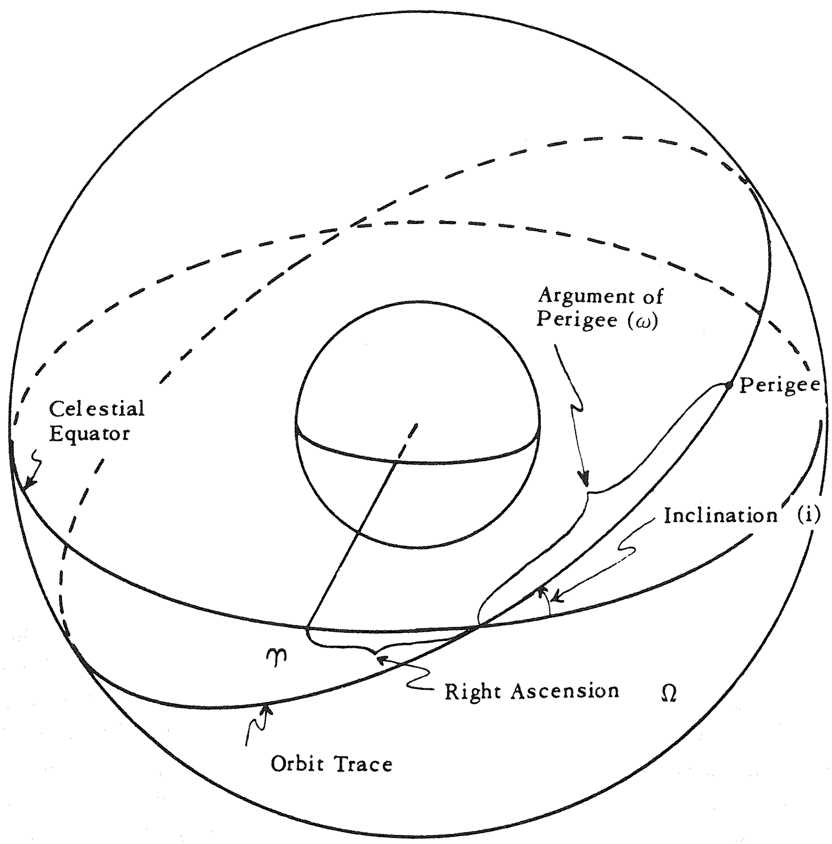


Figure 21. Celestial Sphere.

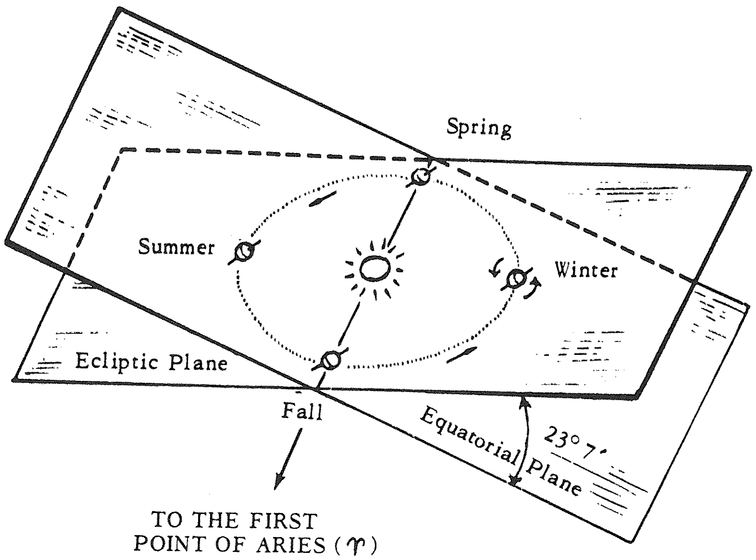


Figure 22. The vernal equinox.

Therefore, the celestial sphere is used to satisfy the requirement for a reference frame. This is a nonrotating sphere of infinite radius whose center coincides with the center of the earth and whose surface contains the projection of the celestial bodies as they appear in the sky (Fig. 21). The celestial equator is a projection of the earth's equator on the celestial sphere. The track of a satellite can be projected on the celestial sphere by extending the plane of the oribt to its intersection with the celestial sphere.

After the center of the system and the celestial equator have been defined, a reference is required as a starting point for position measurements. This point, determined at the instant winter changes into spring, is found by passing a line from the center of the earth through the center of the sun to the celestial equator, and is called the vernal equinox (Fig. 22).

After the references for the coordinate system have been established, the orbit itself must be located. The first item of importance is *right ascension* (Ω) of the ascending node, which is defined as the arc of the celestial equator measured eastward from the vernal equinox to the ascending node (Fig. 21). The ascending node is the point where the projection of the satellite path crosses the celestial equator from south to north. In other words, right ascension of the ascending node is the angle measured eastward from the first point of Aries to the point where the satellite crosses the equator from south to north.

The next item of importance is the angle the path of the orbit makes with the equator. This is the angle of inclination (i), which is defined as the angle that the plane of the orbit makes with the plane of the equator, measured counterclockwise from the equator at the ascending node. *Equatorial* orbits have i = 0°; *posigrade* orbits have i = 0° to 90°; *polar* orbits have i = 90°; and *retrograde* orbits have i= 90° to 180°.

To describe the orbit further, the perigee is located. The angular measurement from the ascending node to the perigee, measured along the path of the orbit in the direction of motion, is called the argument of perigee (ω).

If, in addition to the coordinates of the orbit, a time of either perigee or right ascension of the ascending node is known, along with the eccentricity and the major axis of the orbit, the exact position and velocity of the satellite can be determined at any time. Six quantities (right ascension, inclination, argument of perigee, eccentricity, major axis, and epoch time at either perigee or ascending node) form a convenient grouping of the minimum information necessary to describe the orbital path as well as the position of a satellite at any time. They constitute one set of orbital elements, known as the Breakwell Set of Keplerian Elements.

Orbital Plane

Another interesting facet of earth satellites concerns the orbital plane. There is a relationship between the launch site and the possible orbital planes. This restriction arises from the fact that the center of the earth must be a focus of the orbit and, therefore, must lie in the orbital plane.

The inclination of the orbital plane, i, to the equatorial plane is determined by the following formula: *cos i (inclination) = cos (latitude) sin (azimuth)* where the azimuth is the heading of the vehicle measured clockwise from true north.

As an example, a satellite launched from Cape Kennedy and injected at 30° N on a heading due east (Azimuth 90°) will lie in an orbital plane which is inclined 30° to the equatorial plane.

$$
\begin{aligned}
\cos i &= (\cos \text{latitude})\ (\sin \text{azimuth}) \\
&= \cos 30^\circ \sin 90^\circ \\
\cos i &= \cos 30^\circ \\
i &= 30^\circ
\end{aligned}
$$

It can be deduced from the above that the minimum orbital plane inclination for a direct (no dog leg or maneuvering) injection will be closely defined by the latitude of the launch site. All launch sites, thereby, will permit direct injections at inclination angles from that minimum (the approximate latitude of the launch site) to polar orbits (plus retrograde supplements), provided there were no geographic restrictions on launch azimuth, such as range safety limitations. For example, direct injections from Vandenberg AFB (35°N) would permit inclination angles from about 35° to 145°.

Once the inclination of the orbital plane is defined, the ground track can be discussed.

SATELLITE GROUND TRACKS

The orbits of all satellites lie in planes which pass through the center of a theoretically spherical earth. Each plane intersects the surface of the earth in a great circle (Fig. 23).

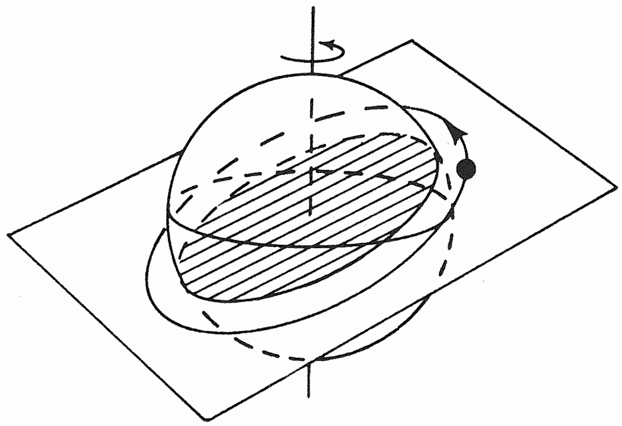


Figure 23. Satellite ground track geometry.

A satellite's ground track is formed by the intersection of the surface of the earth and a line between the center of the earth and the satellite. As the space vehicle moves in its orbit, this intersection traces out a path on the ground below.

There are five primary factors which affect the ground track of a satellite moving along a free flight trajectory. These are:

1. Injection point	
2. Inclination angle	(i)
3. Period	(P)
4. Eccentricity	(ϵ)
5. Argument of Perigee	(ω)

Of the above, the injection point simply determines the point on the surface from which the ground track begins, following orbital injection of the satellite. Inclination angle has been discussed in the previous section and will be treated below in further detail. Period, eccentricity, and argument of perigee each affect the ground track, but it is often difficult to isolate the effect of any one of the three. Therefore, only general remarks regarding the three factors will be made, rather than an intricate mathematical treatment.

If the study of satellite ground tracks is predicated upon a nonrotating earth, the track of a satellite in a circular orbit is easy to visualize. When the satellite's orbit is in the equatorial plane, the ground track coincides with the equator (Fig. 24).

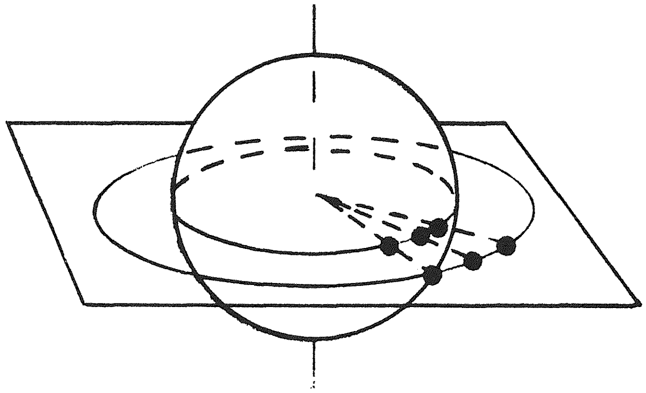


Figure 24. Equatorial track.

If the plane of the orbit is inclined to the equatorial plane, the ground track moves north and south of the equator. It moves between the limits of latitude equal to the inclination of its orbital plane (Fig. 25). A satellite in either circular or elliptical orbit will trace out a path over the earth between these same limits of latitude, determined by the inclination angle. However, the satellite in elliptical orbit will, with one exception, remain north or south of the equator for unequal periods of time. This exception occurs when the major or long axis of the orbit lies in the equatorial plane.

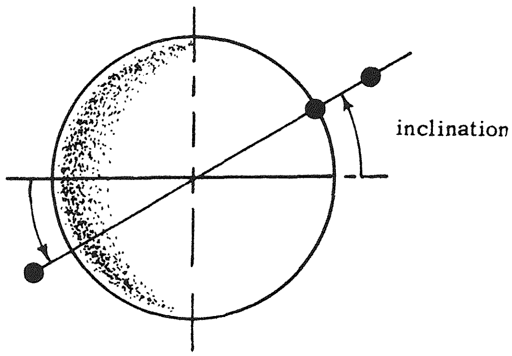


Figure 25. North-South travel limits.

The inclination of an orbit is determined by both the latitude of the vehicle and the direction of the vehicle's velocity at the time of injection or entry into orbit. That is, the cosine of the inclination angle equals the cosine of the latitude times the sine of the azimuth (when the azimuth is measured from north). The minimum inclination which an orbital plane can be made to assume is the number of degrees of latitude at which injection occurs. This minimum inclination occurs when the direction of the vehicle's velocity is due east or west at the time of injection. If the vehicle's direction at injection into orbit is any direction other than east or west, the inclination of the orbital plane will be increased (Fig. 26).

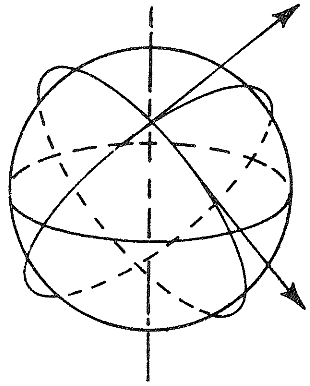


Figure 26. Injection-inclination geometry.

On a flat map of the earth, satellite ground tracks appear to have different shapes than on a sphere. The ground track for a vehicle in an inclined circular or elliptical orbit appears as a sinusoidal trace with North-South limits equal to the inclination of the orbital plane (Fig. 27).

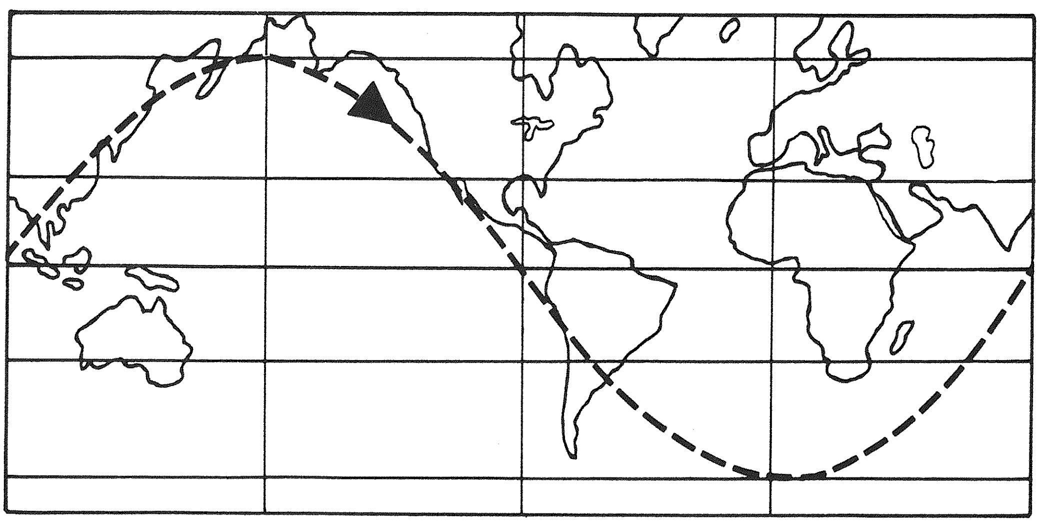


Figure 27. Ground track on flat, non-rotating earth map.

When the earth's rotation is considered, visualizing a satellite's ground track becomes more complex. A point on the equator moves from west to east more rapidly than do points north and south of the equator. Their speeds are the speed of a point on the equator times the cosine of their latitude. Satellites in circular orbit travel at a constant speed. However, when the orbits are inclined to the equator, the component of satellite velocity which is effective in an easterly or westerly direction varies continously throughout the orbital trace (Fig. 28). As the satellite crosses the equator, its easterly or westerly component of velocity is its instantaneous total velocity times the cosine of its angle of inclination.

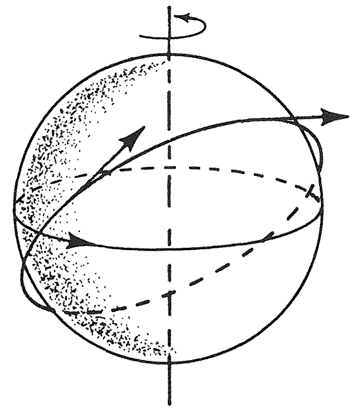


Figure 28. Effective East/West component of satellite velocity.

When it is at the most northerly or southerly portion of its orbit, its easterly or westerly component is equal to its total instantaneous velocity.

In elliptical orbits only the horizontal velocity component contributes to the satellite's ground track. Further complication results because the inertial or absolute speed of the satellite varies throughout the elliptical path (Fig. 29).

Because the ground track is dependent upon the relative motion between the satellite and the earth, the visualization of ground tracks becomes quite complicated. Earth rotation causes each successive track of a satellite in a near earth orbit to cross the equator west of the preceding track (Fig. 30). This westerly regression is equal to the period of the satellite times the rotational speed of the earth. The regression is more clearly seen if angular speed is considered. The earth's angular speed of rotation is 15°/hour. The number of degrees of regression (in terms of a shift in longitude) can be determined by multiplying the period of the satellite by 15°/hour, the angular speed of the earth. If the altitude of a satellite is increased, thereby increasing the time required to complete one revolution in the orbit, the distance between successive crossings of the equator increases.

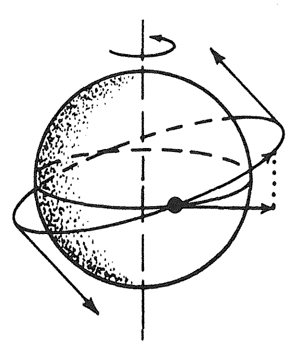


Figure 29. Variation of velocity magnitude in an elliptical orbit.

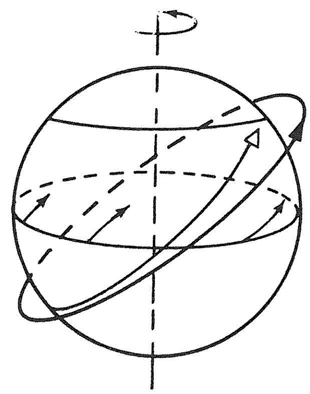


Figure 30. Ground track regression due to earth rotation.

Again, using a flat map of the earth, the track of a satellite in circular orbit (with a period of less than 20 hrs) appears as a series of sinusoidal traces, each successively displaced to the west (Fig. 31).

The ground track of a satellite in elliptical orbit results in a series of irregular traces on a flat map which have one lobe larger than the other. The lobes are compressed by an amount which depends on orbital time, are altered in shape by the combined factors (inclination, eccentricity, period, and location of perigee), and successive traces are displaced to the west (Fig. 32).

Some satellites follow orbits that have particularly interesting ground tracks. A satellite with a 24-hour period of revolution is one such case. If this satellite is in a circular orbit in the equatorial plane, it is often referred to as a *synchronous*

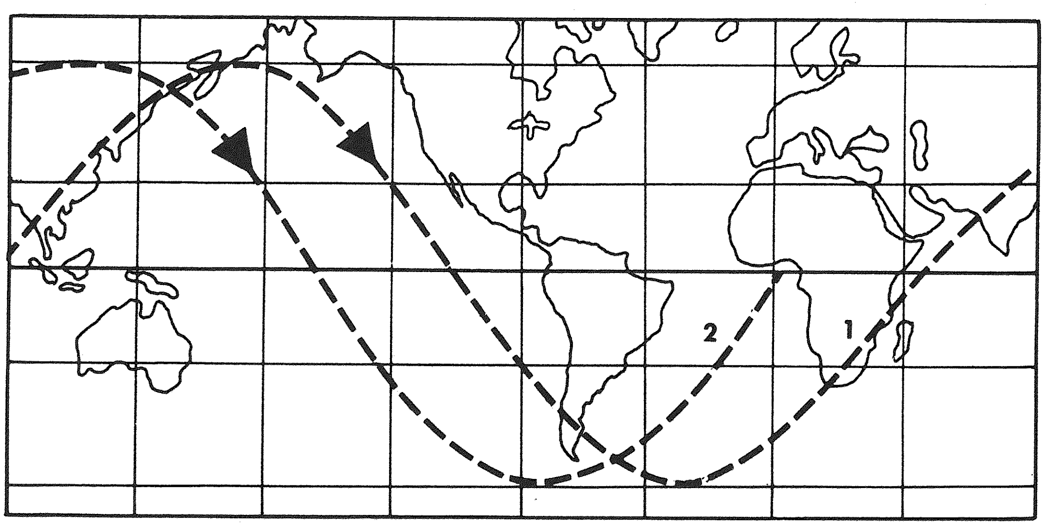


Figure 31. Westward regression of sinusoidal tracks.

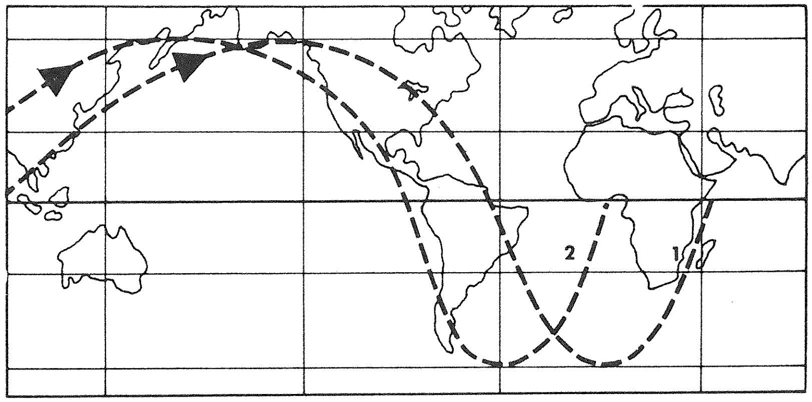


Figure 32. Westward regression of irregular tracks.

satellite; its trace is a single point. If it orbits in the polar plane, it will complete half of its orbit while the earth is rotating halfway about its axis. The result is a trace which crosses a single point on the equator as the satellite crosses the equator heading north and south. The complete ground track forms a figure eight. If the plane of the circular orbit is inclined at smaller angles to the equator, the figure eights are correspondingly smaller (Fig. 33).

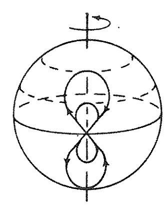


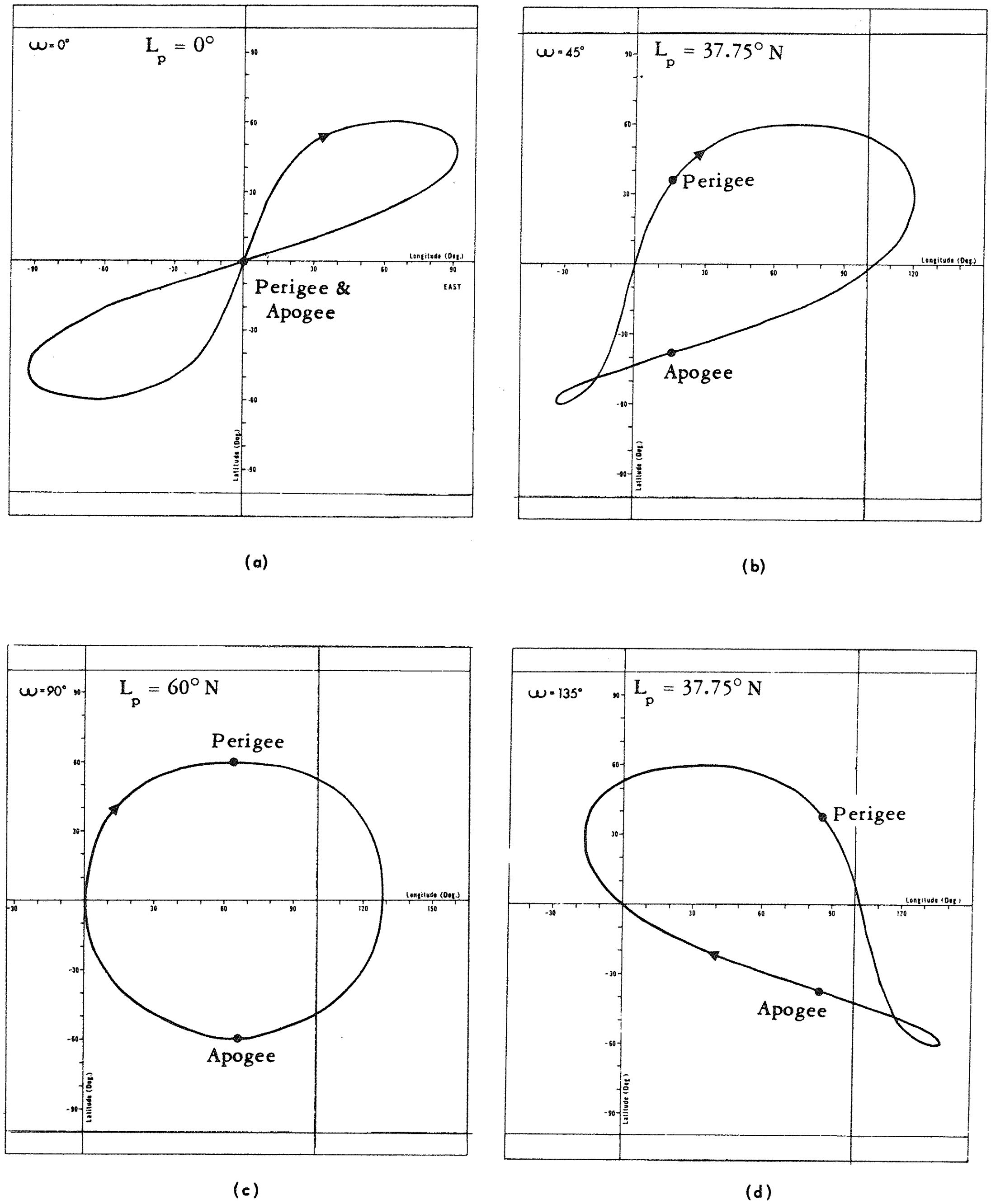
Figure 33. Figure eights for inclinations less than polar.

The shape of the figure eight may be altered by placing a satellite in an elliptical flight path. The eccentricity of the ellipse changes the relative size of the loops of the figure eight. If eccentricity and inclination are fixed, then changing the location of perigee will vary the shape and orientation of the figure eight as shown in Fig. 34. The longitude of perigee is, of course, determined by the conditions and geographical location of injection into the 24-hour orbit. However, the latitude of perigee is fixed by the inclination of the orbital plane (i), and the argument of perigee (ω):

$$\text{Sin (latitude of perigee)} = \sin i \sin \omega$$

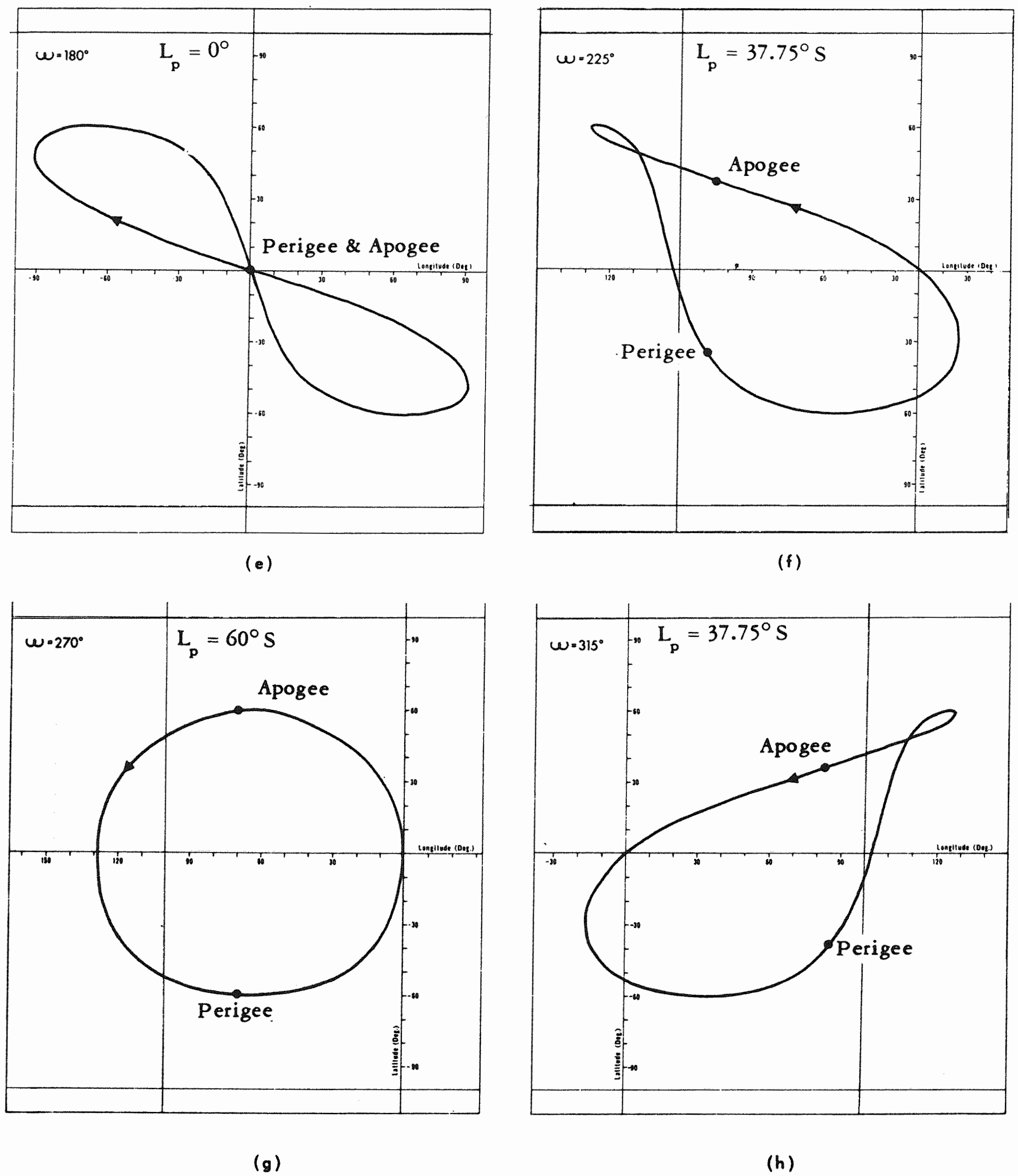
Apogee is located on the same meridian as perigee and is at the same degree of latitude but in the opposite hemisphere.

Circular ground tracks similar to the one shown in Fig. 34(g) have been proposed in certain navigation satellite concepts. In this case with apogee in the northern hemisphere, a multiple satellite system using this type of ground track might be used for supersonic aircraft navigation in the North Atlantic.



Eccentricity (ϵ) = .6
Inclination (i) = 60°
Argument of Perigee (ω) = As Stated

Figure 34. Variation of elliptical, 24-hour track with movement of perigee.



Eccentricity (ϵ) = .6
Inclination (i) = 60°
Argument of Perigee (ω) = As Stated

Figure 34. Variation of elliptical, 24-hour track with movement of perigee, continued.

If a satellite is in an orbit with a period much greater than the earth's twenty four hour period of rotation, the satellite appears as a point in space under which the earth rotates. If the orbit is in the equatorial plane, the trace is a line on the equator moving to the west. If the orbit is inclined to the equator, its earth track will appear to be a continuous trace wound around the earth like a spiral between latitude limits equal to its angle of inclination (Fig. 35). An example of this phenomena is the ground track of the moon.

It should now be clear that there is an almost limitless variety of satellite ground tracks. To obtain a particular track, it is only necessary that the proper orbit be selected. If changes are made in the inclination of an orbital plane to the equator, if its period is varied, if the eccentricity is controlled, or, if the location or perigee is specified, many different ground tracks can be achieved.

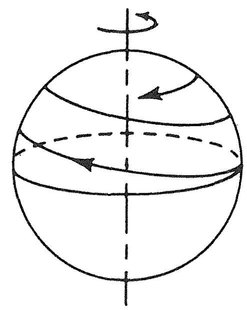


Figure 35. Track for inclined orbit with period greater than one day.

SPACE MANEUVERS

One characteristic of satellites is that their orbits are basically stable in inertial space. This stability is often an advantage, but it can also pose problems. Space operations such as resupply, rendezvous, and interception may require that the orbits of space vehicles be changed. Such changes are usually changes in orbital altitude, orbital plane (inclination), or both. In this section some methods of maneuvering in space will be reviewed.

Altitude Change

When a satellite or space vehicle is to have its orbit changed in altitude, additional energy is required. This is true whether the altitude is increased or decreased. The classic example of changing the orbital altitude of a satellite is the HOHMANN TRANSFER.

The Hohmann transfer is a two-impulse maneuver between two circular, coplanar orbits. For most practical problems, this method uses the least amount of fuel and is known as a minimum energy transfer. The path of the transfer follows an ellipse which is cotangential to the two circular orbits (Fig. 36).

In accomplishing a Hohmann transfer, two applications of thrust are required. Each application of thrust changes the speed of the vehicle and places it into a new orbit. Obviously both the direction and magnitude of the velocity change, Δv,

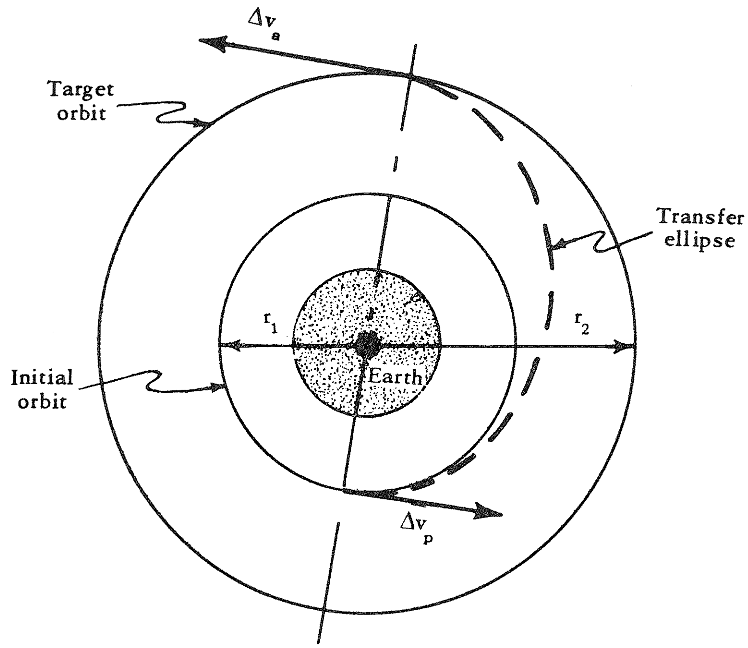


Figure 36

must be accurately controlled for a precise maneuver. If an increase in altitude is desired, the point of departure becomes the perigee of the transfer ellipse; the point of injection to the higher circular orbit becomes the apogee of the transfer ellipse. (For the transfer ellipse, $2a = r_1 + r_2$.) To lower altitude, the reverse is true. The point of departure will be the apogee of the transfer ellipse.

In general, the process for determining the total increment of velocity, Δv, required to complete a Hohmann transfer can be divided into seven steps.

(1) Determine the velocity the vehicle has in the initial orbit.

$$V_{c1} = \sqrt{\frac{\mu}{r_1}}$$

(2) Determine the velocity required at the initial point in the transfer orbit.

$$v_1 = \sqrt{\frac{2\mu}{r_1} - \frac{\mu}{a}}$$

(3) Solve for the vector difference between the velocities found in steps 1 and 2.

(4) Find the velocity the vehicle has at the final point in the transfer ellipse.

$$v_2 = \sqrt{\frac{2\mu}{r_2} - \frac{\mu}{a}} \text{ or } v_2 = \frac{v_1 r_1}{r_2}$$

(5) Compute the velocity required to keep the vehicle in the final orbit.

$$V_{c2} = \sqrt{\frac{\mu}{r_2}}$$

(6) Find the vector difference between the velocities found in steps 4 and 5.

(7) Find the total Δv for the maneuver by adding the Δv from step 3 to the Δv from step 6.

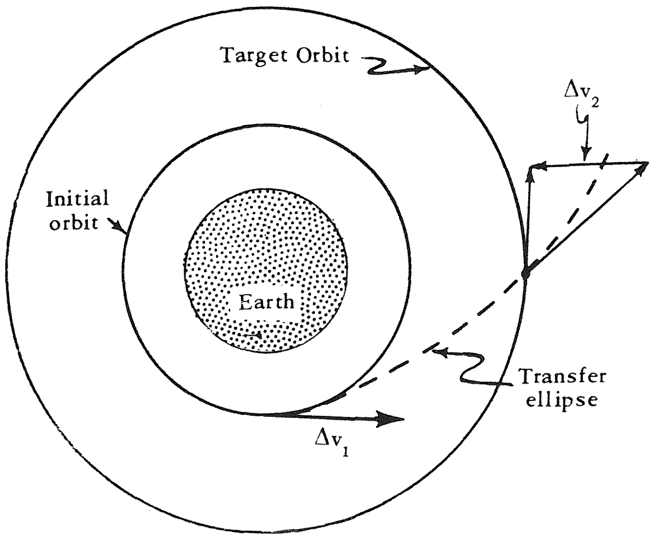


Figure 37

From a practical point, once the required Δv is known, the amount of propellant required for the maneuver can be computed from

$$\frac{\Delta v}{I_{sp}\, g} = \ln\left(\frac{W_o}{W}\right) = \ln \text{ mass ratio}$$

as discussed in Chapter 3.

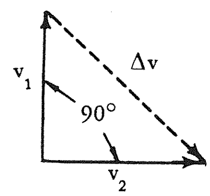
There are, of course, other ways to accomplish an altitude change. One such method is the Fast Transfer, useful when time is a factor.

In the Fast Transfer, the transfer ellipse is not cotangential to the final orbit but crosses it at an angle (Fig. 37). Again the Δv is applied in two increments, but Δv_2, applied at the intersection of the transfer ellipse and the target orbit, must achieve the desired final velocity *in the proper direction.* The steps for calculating the required Δv are similar to those for the Hohmann transfer, noting that velocity differences must be treated as vectorial quantities. For the same altitude change the fast transfer requires more Δv.

Other methods of changing altitudes are discussed in other texts on astronautics. The procedures are similar to those discussed here. The magnitude and direction of the velocity vector remains the critical factor.

Plane Change

Changing the orbital plane of a satellite also requires the expenditure of energy. This is apparent if the vector diagram representing two circular orbits of the same altitude is examined, the only difference being in their inclinations:



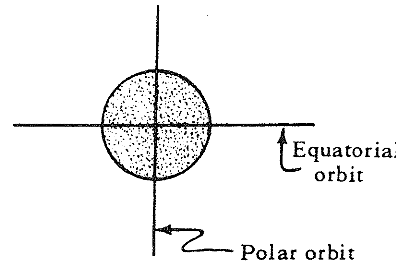
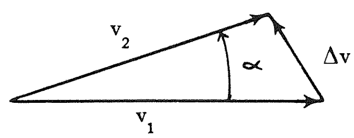


Figure 38

In this case, the orbital speeds, v_1 and v_2, are equal except that they are 90° to each other. Transferring from the polar orbit to the equatorial orbit would require the Δv represented by the dashed arrow. For a 90° plane change (an extreme case) the Δv exceeds the existing orbital speed. It should be noted that *a change from one plane to another can only be accomplished at the intersection of the two planes.*

The Δv required to change the orbital plane any specified amount can be determined by examining the vectors involved. The problem is solvable by use of the Law of Cosines. For example, if v_1 represents the existing orbital velocity, v_2 the final orbital velocity, and α the desired plane change angle,* then:



$$\Delta v^2 = v_1^2 + v_2^2 - 2v_1 v_2 \cos \alpha$$

Figure 39

If only the plane is to be changed, then $v_1 = v_2$ and the problem is simplified. But, if altitude (or eccentricity) is also to be changed, v_1 and v_2 will not be equal.

The amount of Δv required to accomplish a plane change is, of course, dependent upon the amount of change desired. It is also a function of the altitude at which the change is made.

* See appendix B.

Less Δv is required to make a plane change at high altitudes than at low altitudes because the orbital speed of the vehicle is less at higher altitudes. In other words, it is more economical, in terms of propellant required, to make plane changes where the speed of the satellite is low—at apogee, or at high altitudes.

Combined Maneuvers

If a requirement exists to perform both a plane change and an altitude change, some economy will result if the operations are combined.

The problem of combining a plane and altitude change is solved quite simply by considering the vector diagram. For example, it is desired to change the altitude of a vehicle from 100 NM circular orbit to 1500 NM circular orbit with a plane change of 10°. Recognizing that the plane change is more economically made at altitude, the plan is to combine the plane change with injection from the Hohmann transfer ellipse into the 1500 NM circular orbit. A typical Hohmann altitude change is initiated at a point of intersection of the two planes by increasing the vehicle's speed.

At the apogee of the transfer ellipse (1500 NM altitude) the vehicle's speed is 19,800 ft/sec. The required circular speed is 21,650 ft/sec. The complete problem, at apogee, looks like this:

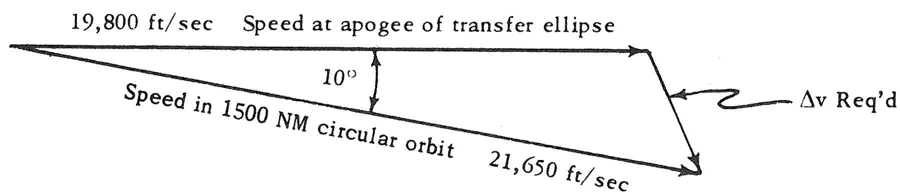


Figure 40

The Δv required for the combined maneuver is calculated by use of the Law of Cosines and is Δv = 4,055 ft/sec. It is also necessary to compute the angle at which the Δv is to be applied. This calculation may be accomplished with the Law of Sines.

PERTURBATIONS

When the orbit of an artificial satellite is calculated by use of the assumptions given thus far, it will vary slightly from the actual orbit unless corrections are made to take care of outside forces. These outside forces, known as perturbations, cause deviations in the orbit from those predicted by two-body orbital mechanics. In order to get a better understanding of how the satellite's actual orbit is going to behave, one must consider the following additional factors:

(1) The earth is not the only source of gravitational attraction on the satellite since there are other gravitational fields (principally of the sun and moon) in space. This effect is greatest at high altitudes (above 20,000 NM).

(2) The earth is not a spherically homogeneous mass but has a bulge around the equatorial region. This additional mass causes the gravitational pull on the satellite *not* to be directed toward the center of the earth. This is the major perturbation effect at medium altitudes (between 300 NM and 20,000 NM).

(3) The earth has an atmosphere which causes drag. This effect is most significant at low altitudes (below 300 NM).

Third Body Effects

One cause of perturbations is the introduction of one or more additional bodies to the system creating a problem involving three or more bodies. Figure 41 shows an exaggerated perturbation, referred to as a hyperbolic encounter. Initially, the satellite is in orbit 1 about an attracting body such as the earth.

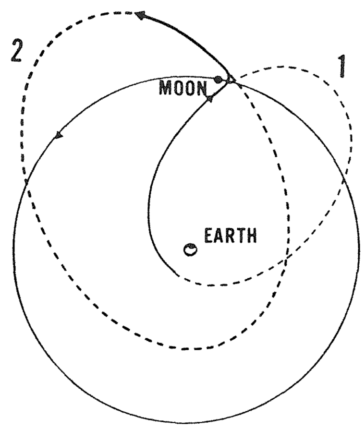


Figure 41. Hyperbolic encounter.

As the satellite approaches the moon, the gravitational influence of the moon dominates, and the center of the moon becomes the focus instead of the center of the earth. Since the vehicle approaches the moon with more than escape velocity, it must leave the moon's sphere of gravitational influence with greater than escape velocity. This means that the vehicle's velocity with respect to the moon is greater than that required to escape. Therefore, for the short time that the moon is the attracting body, the vehicle is on a hyperbolic path with respect to the moon. When the satellite leaves the sphere of influence of the moon, it switches back to the earth's sphere of influence and goes into a new elliptical orbit about the earth. The next time the satellite returns to this region, the moon will have moved in its orbit, and the satellite, therefore, will now maintain orbit 2. A hyperbolic encounter is a method of changing the energy level of a satellite. By proper positioning, it could be used to increase or decrease the energy of a space vehicle. Of course, the change in energy of the vehicle is offset by the change in energy of the second body (the moon in the case illustrated).

Effects of Oblate Earth

Another cause of perturbations is the bulge of the earth at the equator sometimes called the earth's oblateness. The effect of this oblateness on the satellite can be seen if we imagine the earth to be made up of a sphere which has an added belt of mass wrapped around the equatorial region. As shown in Fig. 42, the primary gravitational attraction F, directed to the center of the earth, will now be "disturbed" by the much smaller but nevertheless significant attractions F_1 and F_2 directed toward the near and far sides of the equatorial bulge. With r_1 smaller than r_2, F_1 will be larger than F_2, and the resultant force obtained by combining F, F_1, and F_2 will now no longer point to the center of the earth but will be deflected slightly toward the equator on the near side. As the satellite moves in its orbit the amount of this deflection will change depending on the vehicle's relative position and proximity to the equatorial region.

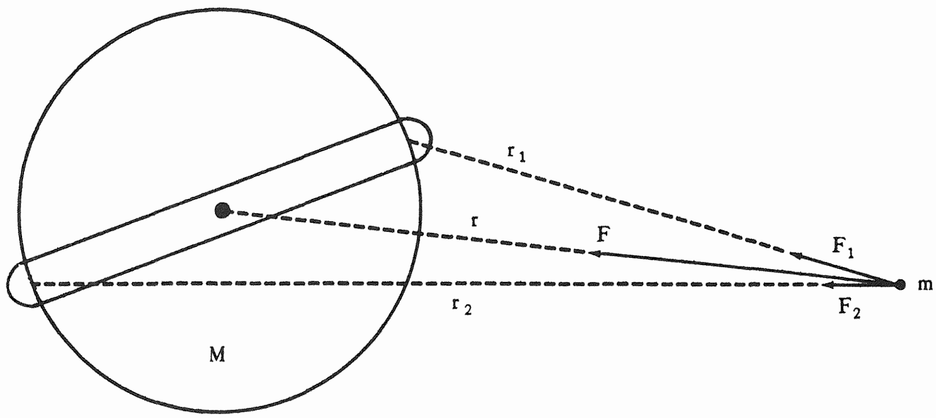


Figure 42. Deviation in force vector caused by the oblateness of the earth

Two perturbations which result from this shift in the gravitational force are:

(1) Regression of the nodes.
(2) Rotation of the line of apsides (major axis) or rotation of perigee.

Regression of the nodes is illustrated in Fig. 43 as a rotation of the plane of the orbit in space. The resulting effect is that the nodes, both ascending and descending, move west or east along the equator with each succeeding pass. The direction of this movement will be opposite to the east or west component of the satellite's motion. Satellites in the posigrade orbit (inclinations less than 90°) illustrated in Fig. 43 have easterly components of velocity so that the nodal regression in this case is to the west. The movement of the nodes will be reversed for retrograde orbits since they always have a westerly component of velocity. Figure 44 shows why, for a vehicle traveling west to east, the regression of the nodes is toward the west. The original track from A to B would cross the equator at Ω_1. Simplifying the effect of the equatorial bulge to a single

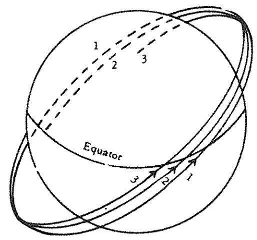


Figure 43. Regression of the nodes.

impulse at point E, the track is moved so that it crosses the equator at Ω_2. At point F the simplified effect of the bulge is a single impulse down, changing the orbital path line to the line FD, which would have crossed the equator at Ω_3. This effect, regression of the nodes, is more pronounced on low-altitude satellites than high-altitude satellites. In low altitude, low inclination orbits the regression rate may be as high as 9° per day. Fig. 45 shows how the regression rate changes for circular orbits at various altitudes and inclination angles. Note that nodal regression is zero in the polar orbit case. It has no meaning in equatorial orbits.

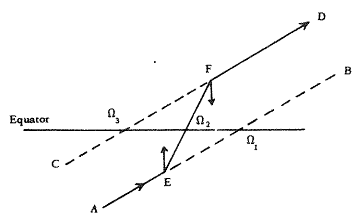


Figure 44. Regression of nodes toward the west when vehicle is traveling west to east.

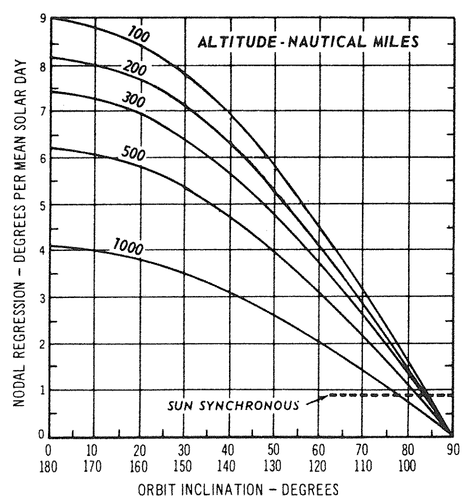


Figure 45. Nodal regression rate per day for circular orbits.

Satellites requiring sun synchronous orbits (for photography or other reasons) are an example of how regression of the nodes can be used to practical advantage in certain situations. In Fig. 46 the satellite is injected into an orbit passing over the equator on the sunlit side of the earth at local noon (the sun overhead). This condition initially aligns the orbital plane so that it contains a line between the earth and sun. The altitude of the near circular orbit determines the angle of inclination required in order to maintain the sun synchronous nodal regression rate of approximately one degree per day (360°/year). In Fig. 45 note the required inclination angles of approximately 95° to 105° for the altitudes shown. If inclination angle and orbital altitude have been chosen correctly, then regression of the nodes will rotate the plane of the orbit (change the angle of right ascension) through 90° every three months as shown in Fig. 46. Thus, we see that this perturbing phenomenon due to the earth's equatorial bulge maintains the desired angle of right ascension which, along with the angle of inclination, orients the orbital plane within the celestial sphere. Therefore, as the earth moves in its orbit the desired orientation for best photography is maintained without using propellant.

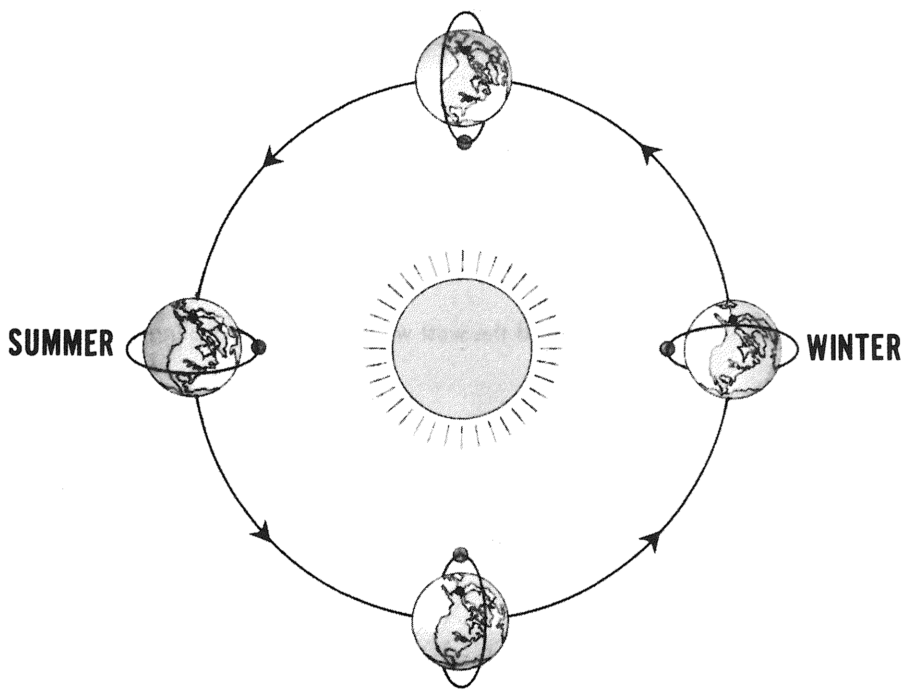


Figure 46. Sun-synchronous orbit.

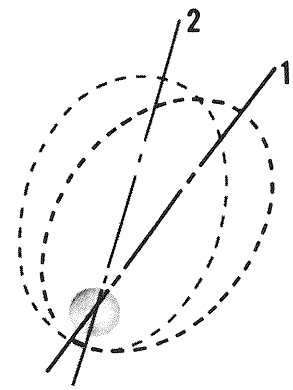


Figure 47. Earth's equatorial bulge changes the argument of perigee.

Rotation of the apsidal line is shown in Figure 47. (The apsidal line is the line joining apogee and perigee—the major axis.) The cause of this perturbation due to oblateness is difficult to visualize; however, the result is that the trajectory rotates within the orbital plane about the occupied focus. This rotation has the effect of shifting the location of perigee, thus changing the argument of perigee (Figure 21).

This rate of change in the argument of perigee is a function of satellite altitude and inclination angle. At inclinations of 63.4° and 116.6° the rate of rotation is zero. Figure 48 illustrates how the apsidal rotation rate varies with inclination angle for orbits with a 100 NM perigee and different apogee altitudes.

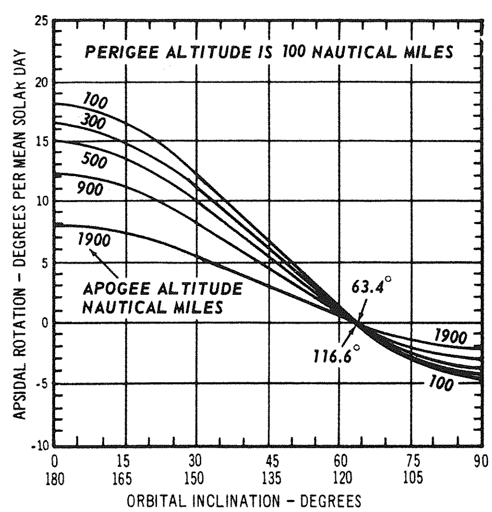


Figure 48. Apsidal rotation rate per day for orbits with 100 NM perigee altitude.

Drag Effects

Drag on a satellite will cause a decrease in eccentricity, a decrease in the major axis, and a rotation of the apsidal line. Figure 49 illustrates the change in eccentricity and major axis. The original orbit is represented by the curve A_1 A_2 A_1, with the focus at F, and a major axis equal to line segment A_1 A_2, Assuming that the drag is concentrated near perigee, the speed at A_1 eventually will diminish to circular speed, and the new path will be A_1 A_3 A_1, with the major axis decreasing to line segment A_1 A_3. After the orbit has decayed to approximately zero eccentricity (a circle), further decay will result in a nearly circular spiral with ever-decreasing radius.

Some other causes of perturbations are electromagnetic forces, radiation pressures, solar pressures, and gas-dynamic forces.

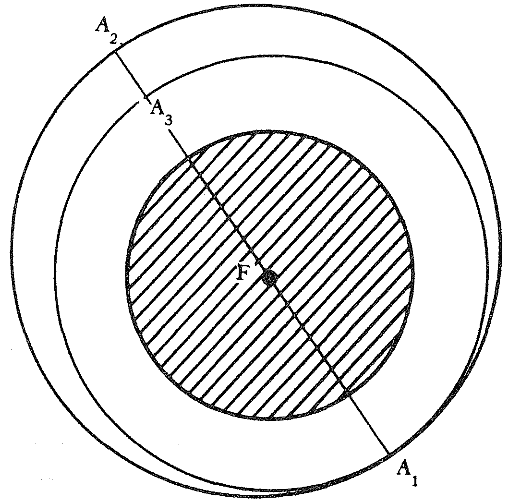


Figure 49. Decrease in the eccentricity of a satellite orbit caused by drag.

THE DEORBITING PROBLEM

The general operation of moving a body from an earth orbit to a precise point on the surface of the earth is a difficult problem. The orbit may be circular or elliptical, high or low, and inclined at various degrees to the rotational axis of the earth. The deorbiting maneuver may be the lofted, depressed, or retro approach which is the only method discussed in this section. Time of flight is a major parameter in deorbiting and presents one of the more difficult of mathematical problems. But, the theory developed here is sufficiently general in scope that it may be modified to solve most problems. The complexity of the problem is reduced in the following simplified illustration of deorbiting from a circular, polar orbit from a point over the North Pole, by assuming no atmosphere.* The approach and solution will permit impacting any earth target, provided fuel for retrothrust is no limitation.

* The student may consider the effect of the atmosphere by using the radius of reentry rather than the radius of the earth, and by then considering the range and time of flight for a specific reentry body. The solution may also be modified to consider elliptical orbits and inclined orbits.

Consider a target at 60° north latitude. If the earth could be stopped from rotating (see Fig. 50)*, with the target in the plane of the circular orbit, then it only would be necessary to apply a retrovelocity (Δv_1) to the orbital body so it would traverse the ellipse shown (arc 1-4) and impact the target. The reentry velocity magnitude (v_b), equal to the circular speed (v_c) minus the retrovelocity (Δv_1), would be necessary at apogee (1) to impact any target on the 60° north parallel. But, the direction of v_b would have to be oriented with respect to the earth's axis so that it would lie in the plane formed by the earth's axis and the predicted position of the target.

Since the earth is rotating at a constant speed, the time of flight (t_b) of the orbital object must be known in order to predict where the target will be at impact. For example, when the orbital vehicle is at Point 1 over the North Pole, the target is at Point 2. During the descent of the payload, however, the target moves to Point 3. In order to predict Point 3, t_b must be known.

Then, if the magnitudes of v_c, v_b, and α, the angle between these two vectors, are known, the Law of Cosines can be used to determine Δv_2. Application of Δv_2 to v_c insures that v_b will have the proper magnitude and direction so that the object will impact a selected target on a rotating earth.

Deorbiting Velocity

Looking first at the geometry of the problem (Fig. 51), notice that the total problem lies in one plane. Basically, the radius of the circular orbit (r_c) and the latitude of the target (L) are known. Since the original circular orbit and the bombing transfer ellipse are coplanar and cotangential, the problem begins as a Hohmann transfer. To determine Δv, the rearward velocity increment which must be applied to cause the object to impact the target can be computed from $\Delta v = v_c - v_b$, if v_c and v_b are known.

$$v_c = \sqrt{\frac{\mu}{r_c}}$$

$$v_b = \sqrt{\frac{2\mu}{r_a} - \frac{\mu}{a}}$$

$$\text{Since } r_a = a + c = a\,(1 + \epsilon),\ a_b = \frac{r_a}{1 + \epsilon_b}$$

$$\text{Then } v_b = \sqrt{\frac{2\mu}{r_a} - \mu\,\frac{(1 + \epsilon_b)}{r_a}} = \sqrt{\frac{\mu}{r_a}\,(1 - \epsilon_b)}$$

Write the general equation of the conic as $k\epsilon = r\,(1 + \epsilon \cos \nu)$ and evaluate at the two known points, r_e** and r_a, noting that at impact $\cos \nu = -\cos \theta_t = -\sin L$, and at retrofire $\cos(180°) = -1$.

Then $k\epsilon = r_e\,(1 - \epsilon_b \sin L) = r_a\,(1 - \epsilon_b)$

* 8 figures related to the deorbit problem appear at the end of this section.

** r_e is radius of the earth, but radius of re-entry could be used if desired.

Solving for ϵ_b.

$$\epsilon_b \, (r_a - r_e \sin L) = r_a - r_e$$

$$\epsilon_b = \frac{r_a - r_e}{r_a - r_e \sin L}$$

Note that in order to find ϵ_b and, in turn, the magnitude of v_b, only the *latitude* of the target and the *altitude* of the circular orbit need be known. Suppose there is a satellite in a 500 NM circular polar oribit. When the orbital vehicle arrives directly over the North Pole, it is desired to deorbit an object which will impact at 60° north latitude. Assuming that the earth has no atmosphere and is nonrotating, what retrovelocity is necessary?

$$\epsilon_b = \frac{r_a - r_e}{r_a - r_e \sin L} = \frac{(23.94 \times 10^6 \text{ ft}) - (20.9 \times 10^6 \text{ ft})}{(23.94 \times 10^6 \text{ ft}) - (20.9 \times 10^6 \text{ ft}) \quad \sin 60°}$$

$$\epsilon_b = .520$$

$$v_b = \sqrt{\frac{\mu}{r_a} (1 - \epsilon_b)} = \sqrt{\frac{14.08 \times 10^{15} \text{ ft}^3/\text{sec}^2}{23.94 \times 10^6 \text{ ft}} (1 - .520)}$$

$$v_b = 16{,}790 \text{ ft/sec}$$

$$\Delta v = v_c - v_b = 24{,}240 \text{ ft/sec} - 16{,}790 \text{ ft/sec}$$

$$\Delta v = 7{,}450 \text{ ft/sec}$$

Looking again at the geometry, note that there is a satellite in a 500 NM circular, polar orbit with $v_c = 24{,}240$ ft/sec. A retrovelocity increment, Δv, equal to 7,450 ft/sec was applied. This provided a magnitude $v_b = 16{,}790$ ft/sec so that the object now follows arc 1-2 and impacts on 60° north latitude.

Figure 52 provides a graph of velocity versus latitude to determine the magnitude of v_b. The graph is for *a specific altitude,* for release from over the North Pole, and for no atmosphere. Included also are more general graphs, Figures 53 and 54.

Deorbit Time of Flight

As the bomb falls from apogee, the target moves toward the east due to earth rotation. Its speed is 1520 cos L ft/sec, so the necessity of accurately computing the time to bomb, t_b, is readily apparent.

First, look at the problem in schematic, Fig. 50. Recall that at the time the vehicle is over the North Pole at Point 1, the target is at Point 2. Thus, t_b must be known so that the location of Point 3 can be predicted. If the meridian that Point 3 will be on, and the meridian with which the circular orbit coincides at the instant of deorbit are known, then the angle α can be determined by subtraction.

There are several methods for computing t_b from release to impact. Eqn 10, App D could be used to determine t_ψ, the time of flight for a ballistic missile launched from 60° N latitude, with apogee at 500 NM over the North Pole, and impacting at 60° N latitude. This would be the t_ψ to traverse arc 5-1-4 (Fig 50). The actual time to bomb from apogee would be half this amount. The general time of flight method is presented here.

$$t_{1 \rightarrow 2} = \sqrt{\frac{a^3}{\mu}} \; (u_2 - \epsilon \sin u_2) - (u_1 - \epsilon \sin u_1) \qquad \text{(App D, Eqn 7)}$$

Since position 1 corresponds to apogee:

$$u_1 = \pi, \; \sin u_1 = 0$$

$$\text{Then } t_b = t_{1 \rightarrow 2} = \sqrt{\frac{a^3}{\mu}} \; (u_t - \epsilon_b \sin u_t - \pi)$$

$$a = \frac{r_a}{1 + \epsilon_b}$$

$$\cos u_t = \frac{\epsilon_b - \cos \theta_t}{1 - \epsilon \cos \theta_t} = \frac{\epsilon_b - \sin L}{1 - \epsilon_b \sin L}$$

All of these parameters are familiar except u which is the eccentric anomaly (Fig. 55). If a perpendicular is dropped through the target to the major axis of the ellipse, it intersects a circle (with the center at C, and the diameter equal to the major axis of the ellipse) at Point Q. By definition, angle BCQ is u, the eccentric anomaly. The radius of the circular orbit, r_a, is given. Both v_b and ϵ_b can be calculated from formulas given previously.

Working with the same example that was used to illustrate deorbiting velocity, t_b will be calculated from only two known quantities—*the altitude of the satellite and the latitude of the target.* A satellite is in a 500 NM circular, polar orbit. Find the time of flight, t_b, from directly over the North Pole to a target on the 60° north parallel:

$$\epsilon_b = \frac{r_a - r_e}{r_a - r_e \sin L} = \frac{(23.94 \times 10^6 \text{ ft}) - (20.9 \times 10^6 \text{ ft})}{(23.94 \times 10^6 \text{ ft}) - (20.9 \times 10^6 \text{ ft}) \sin 60°}$$

$$\epsilon_b = .520$$

$$v_b = \sqrt{\frac{\mu}{r_a} \; (1 - \epsilon_b)} = \sqrt{\frac{14.08 \times 10^{15} \text{ ft}^3/\text{sec}^2 \; (1 - .520)}{23.94 \times 10^6 \text{ ft}}}$$

$$v_b = 16{,}790 \text{ ft/sec}$$

$$a = \frac{r_a}{1 + \epsilon_b} = \frac{23.94 \times 10^6 \text{ ft}}{1.520} = 15.78 \times 10^6 \text{ ft}$$

$$\cos u_t = \frac{\epsilon_b - \sin L\,.}{1 - \epsilon_b \sin L} = \frac{.520 - \sin 60°}{1 - .520 \sin 60°} = -.637$$

Noting u_t lies between 180° and 270°:

$$u_t = 360° - 129.6° = 230.4° = 4.084 \text{ radians}$$

$$t_b = \sqrt{\frac{a^3}{\mu}}\,[u_t - \epsilon_b \sin u_t - \pi]$$

$$t_b = \sqrt{\frac{(15.78 \times 10^6 \text{ ft})^3}{14.08 \times 10^{15} \text{ ft}^3/\text{sec}^2}}\,[4.084 - .520 \sin 230.4° - 3.1416]$$

$$t_b = 676 \text{ sec} = 11.27 \text{ min}$$

Figure 56 shows t_b in minutes versus a plot of target latitude. Once again it must be recognized that this chart is for a specific orbital altitude and no atmosphere. See also the more general graph, Figure 57.

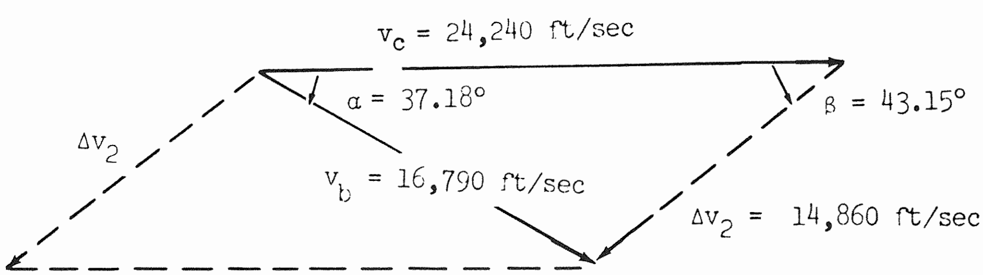
Consider impacting a target at 60° north latitude and 30° east longitude. The plane of the satellite is coincident with the 70° east meridian. This means that, when the satellite is at Point 1 and the target is at Point 2, the angle between the target and orbital plane is 40°. However, *the target is moving.* It moves at:

$$(360°/24 \text{ hrs}) \frac{\text{hr}}{60 \text{ min}} = .25 \text{ deg/min}$$

$$\theta_r = (11.27 \text{ min})\ (.25 \text{ deg/min}) = 2.82°$$

$$\alpha = 40° - 2.82° = 37.18°$$

Now apply the Law of Cosines and determine the Δv_2 which must be applied to v_c in order to impact the target:



$$\Delta v_2 = (v_b^2 + v_c^2 - 2v_b v_c \cos \alpha)^{1/2}$$

$$\Delta v_2 = [(16{,}790 \text{ ft/sec})^2 + (24{,}240 \text{ ft/sec})^2 - 2(16{,}790 \text{ ft/sec} \times 24{,}240 \text{ ft/sec} \cos 37.18°)]^{1/2}$$

$$\Delta v_2 = [2.82 \times 10^8 + 5.88 \times 10^8 - 6.49 \times 10^8]^{1/2}$$

$$\Delta v_2 = [2.21 \times 10^8]^{1/2} = 14{,}860 \text{ ft/sec}$$

Also, the value of angle β must be known so that the proper direction of Δv_2 may be determined. The Law of Sines is preferred for this calculation, although the Law of Cosines can be used.

$$\sin\beta = \frac{v_b \sin\alpha}{\Delta v_2}$$

$$\sin\beta = \frac{16{,}790 \text{ ft/sec} \sin 37.18^\circ}{14{,}860 \text{ ft/sec}}$$

$$\sin\beta = .6827$$

$$\beta = 43.15^\circ$$

From the solution of this simplified object-from-orbit problem, it is apparent that such a calculation is not really simple. Other mathematical approaches and other operational problems are even more difficult. However, the theory presented can be extended to more difficult cases.

Fuel Requirement

It is also of interest to determine the propellant necessary to perform this maneuver. Assuming an I_{sp} = 450 sec (a reasonable figure in the near future) and an initial weight of 10,000 pounds, compute the amount of propellant required, W_p:

$$\ln\left(\frac{W_1}{W_2}\right) = \frac{\Delta v}{I_{sp}g} = \frac{14{,}860 \text{ ft/sec}}{(450 \text{ sec})\ (32.2 \text{ ft/sec}^2)} = 1.025$$

$$\frac{W_1}{W_2} = 2.79$$

$$W_2 = \frac{10{,}000 \text{ lbs}}{2.79} = 3{,}590 \text{ lbs}$$

$$W_p = W_1 - W_2 = 6{,}410 \text{ lbs}$$

This weight of propellant represents $\frac{W_p}{W_1} = \frac{6{,}400}{10{,}000} = 64.1\%$ of the weight in orbit prior to maneuvering.

By restricting the plane change (side range) to α = 7.18 degrees, Δv_2 = 7,875 ft/sec, W_p = 4,190 lbs, and $\frac{W_p}{W_1} = 41.9\%$.

In summary, recall that the altitude of a satellite in a circular orbit and a time when the satellite was over the North Pole were given. A target was selected, and no earth atmosphere was assumed. From the theory, the required velocity for a deorbiting object was calculated, and the position of the target at the time of impact was predicted. With this information and the use of the laws of sines and cosines, the magnitude of retrovelocity and the direction to deorbit on target were calculated. Also, the amount of propellant required was computed.

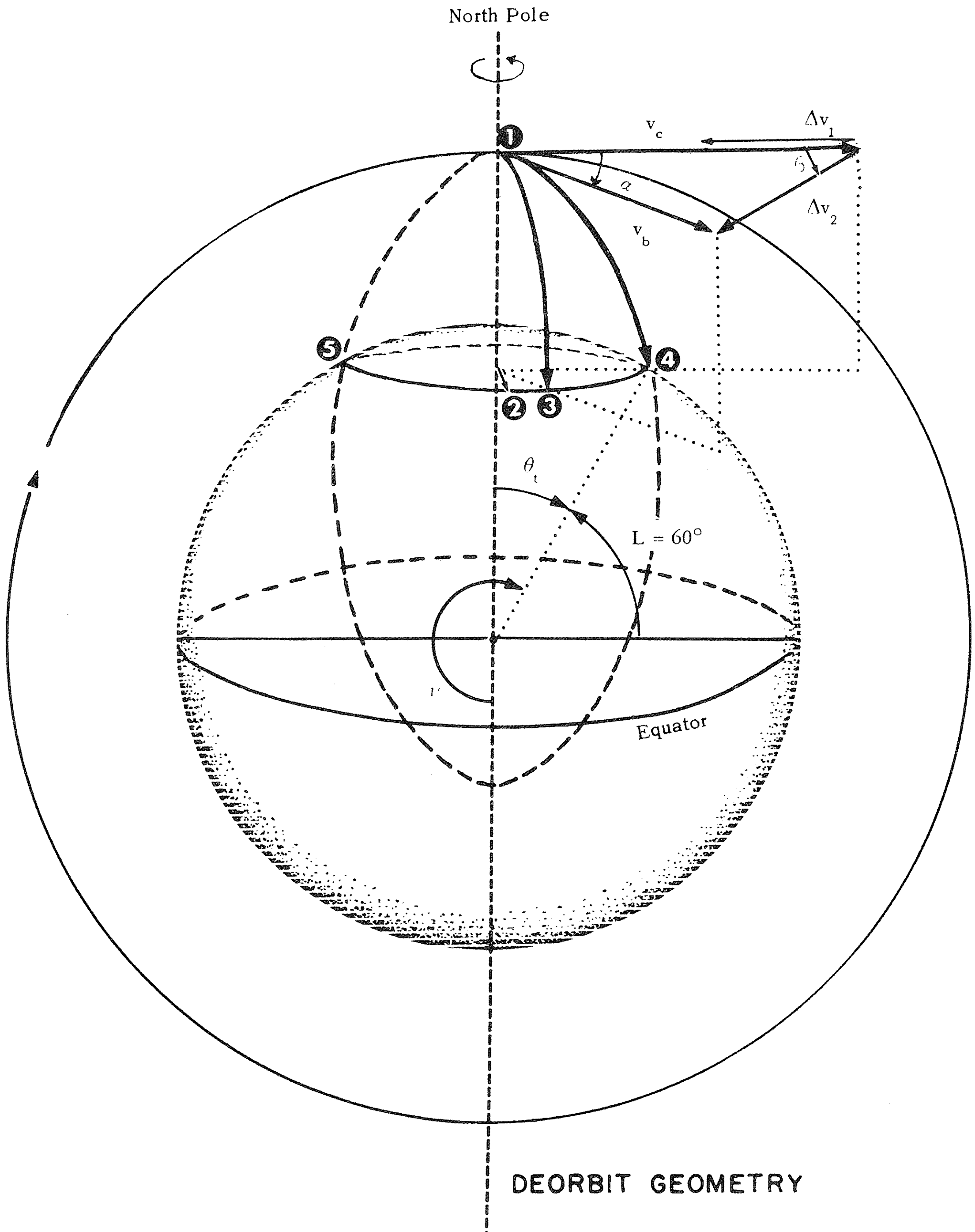

North Pole
v_c
Δv_1
β
α
v_b
Δv_2
1
2
3
4
5
θ_t
L = 60°
ν
Equator
DEORBIT GEOMETRY

Figure 50

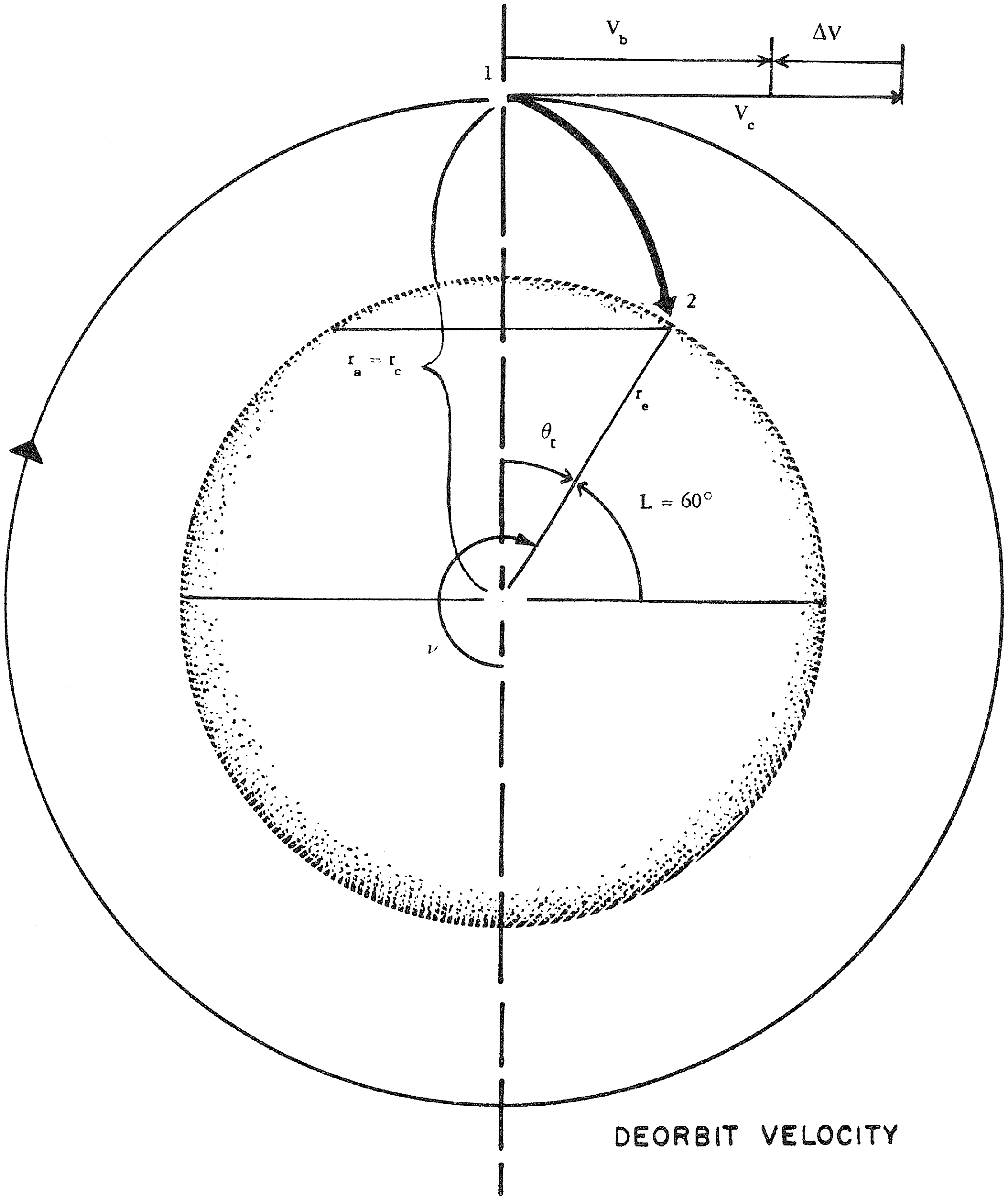


Figure 51

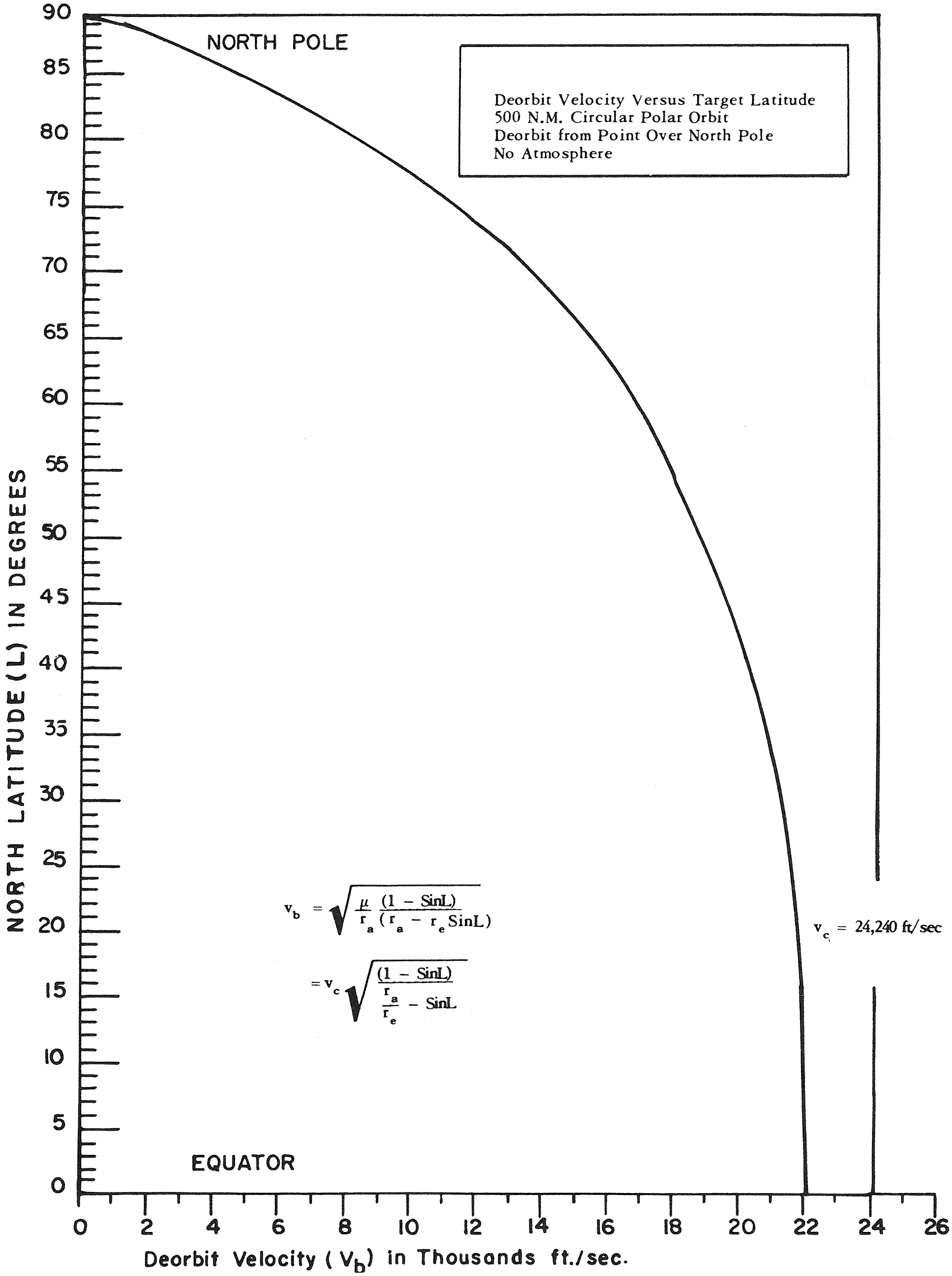

NORTH POLE
Deorbit Velocity Versus Target Latitude
500 N.M. Circular Polar Orbit
Deorbit from Point Over North Pole
No Atmosphere
NORTH LATITUDE (L) IN DEGREES
90
85
80
75
70
65
60
55
50
45
40
35
30
25
20
15
10
5
0
$v_b = \sqrt{\frac{\mu}{r_a} \frac{(1 - \text{Sin}L)}{(r_a - r_e \text{Sin}L)}}$
$= v_c \sqrt{\frac{(1 - \text{Sin}L)}{\frac{r_a}{r_e} - \text{Sin}L}}$
v_c = 24,240 ft/sec
EQUATOR
0
2
4
6
8
10
12
14
16
18
20
22
24
26
Deorbit Velocity (V_b) in Thousands ft./sec.

Figure 52

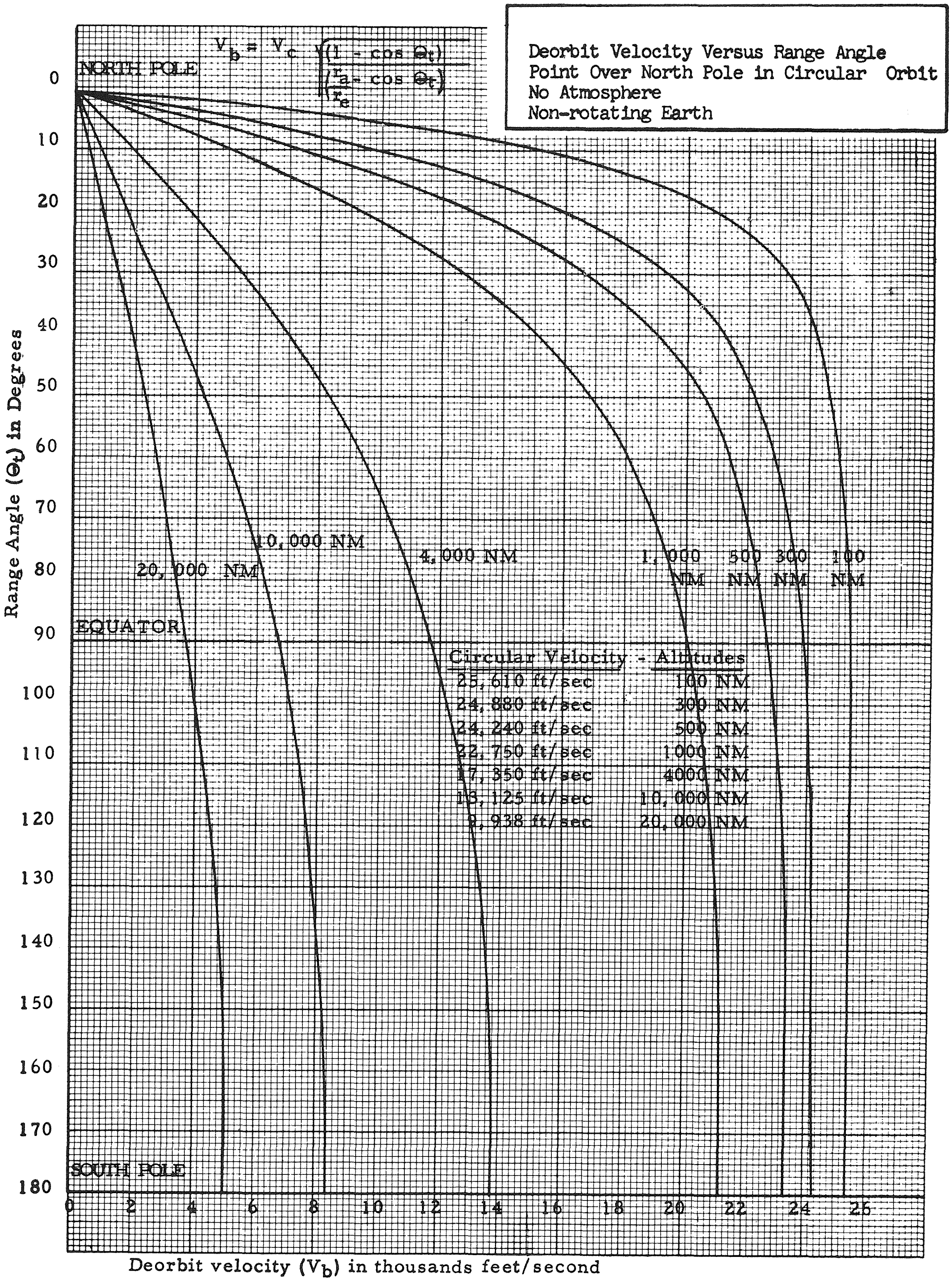

Deorbit Velocity Versus Range Angle
Point Over North Pole in Circular Orbit
No Atmosphere
Non-rotating Earth
$V_b = V_c \sqrt{\dfrac{1 - \cos \Theta_t}{\frac{r_a}{r_e} - \cos \Theta_t}}$
NORTH POLE
EQUATOR
SOUTH POLE
20,000 NM
10,000 NM
4,000 NM
1,000 NM
500 NM
300 NM
100 NM
Circular Velocity - Altitudes
25,610 ft/sec 100 NM
24,880 ft/sec 300 NM
24,240 ft/sec 500 NM
22,750 ft/sec 1000 NM
17,350 ft/sec 4000 NM
13,125 ft/sec 10,000 NM
9,938 ft/sec 20,000 NM
Range Angle (Θ_t) in Degrees
0 10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160 170 180
0 2 4 6 8 10 12 14 16 18 20 22 24 26
Deorbit velocity (V_b) in thousands feet/second

Figure 53

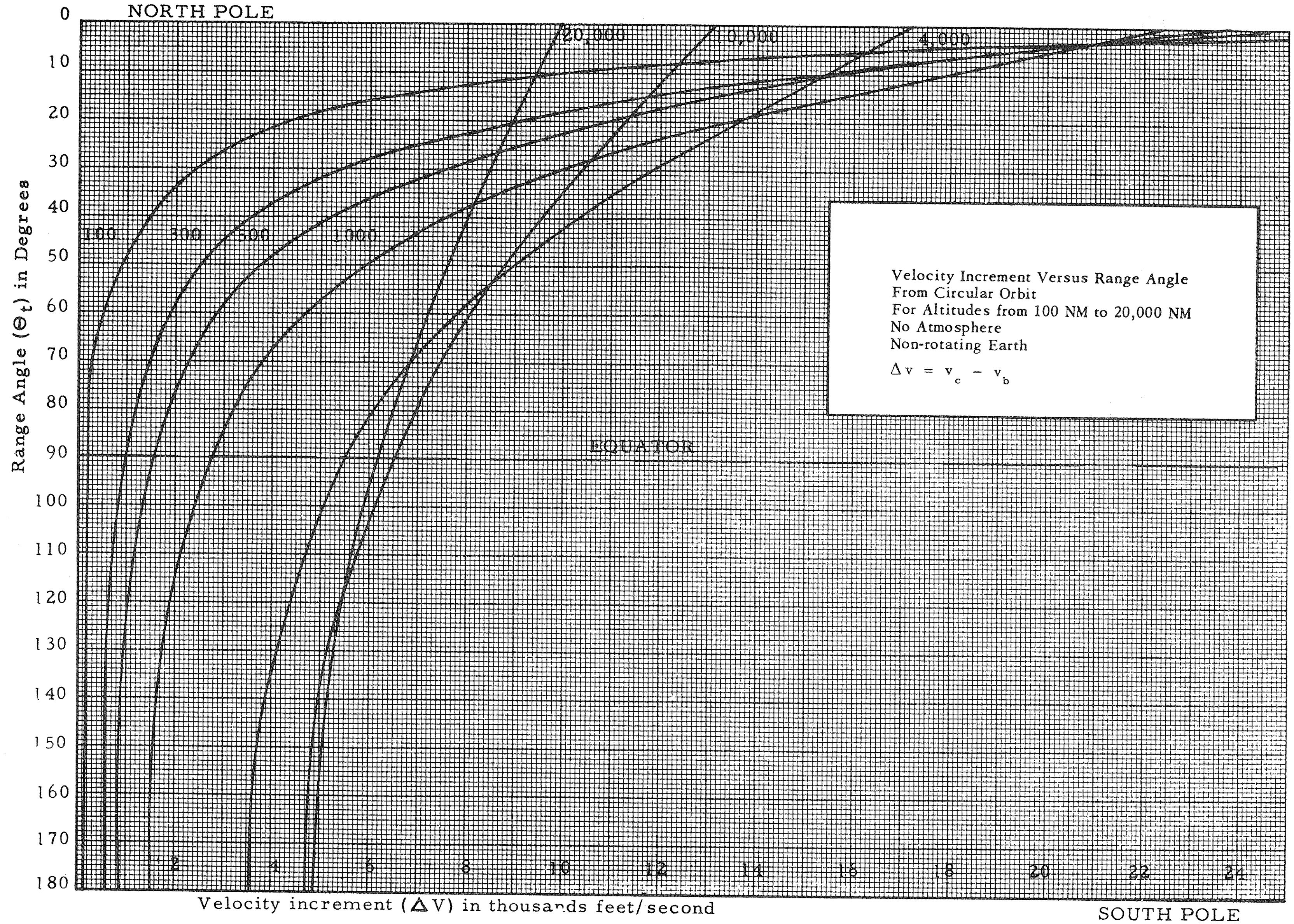

NORTH POLE
SOUTH POLE
EQUATOR
Range Angle (Θ_t) in Degrees
Velocity increment (ΔV) in thousands feet/second
0
10
20
30
40
50
60
70
80
90
100
110
120
130
140
150
160
170
180
2
4
6
8
10
12
14
16
18
20
22
24
100
300
500
1000
20,000
10,000
4,000
Velocity Increment Versus Range Angle
From Circular Orbit
For Altitudes from 100 NM to 20,000 NM
No Atmosphere
Non-rotating Earth
$\Delta v = v_c - v_b$

Figure 54

TIME OF FLIGHT GEOMETRY

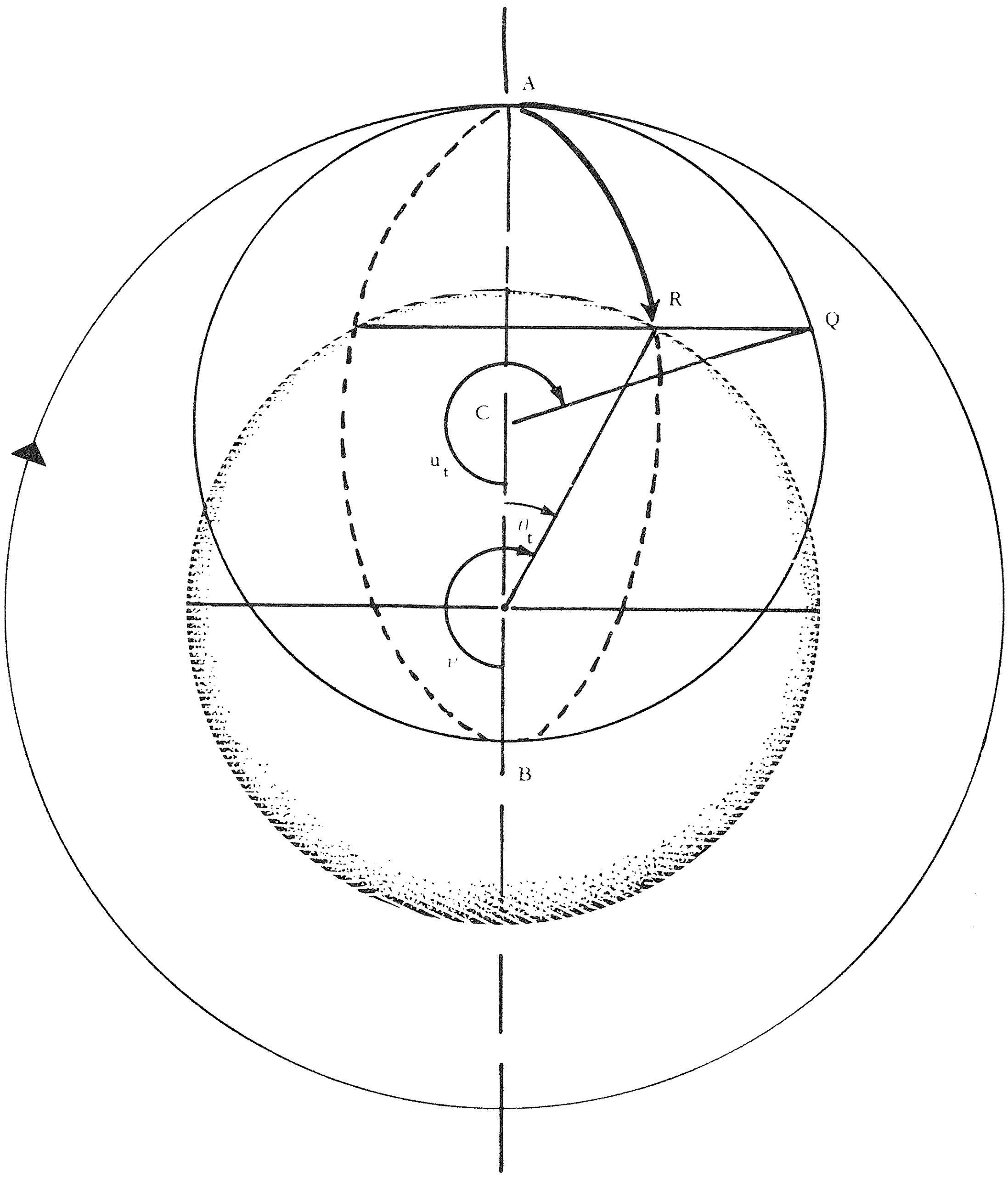
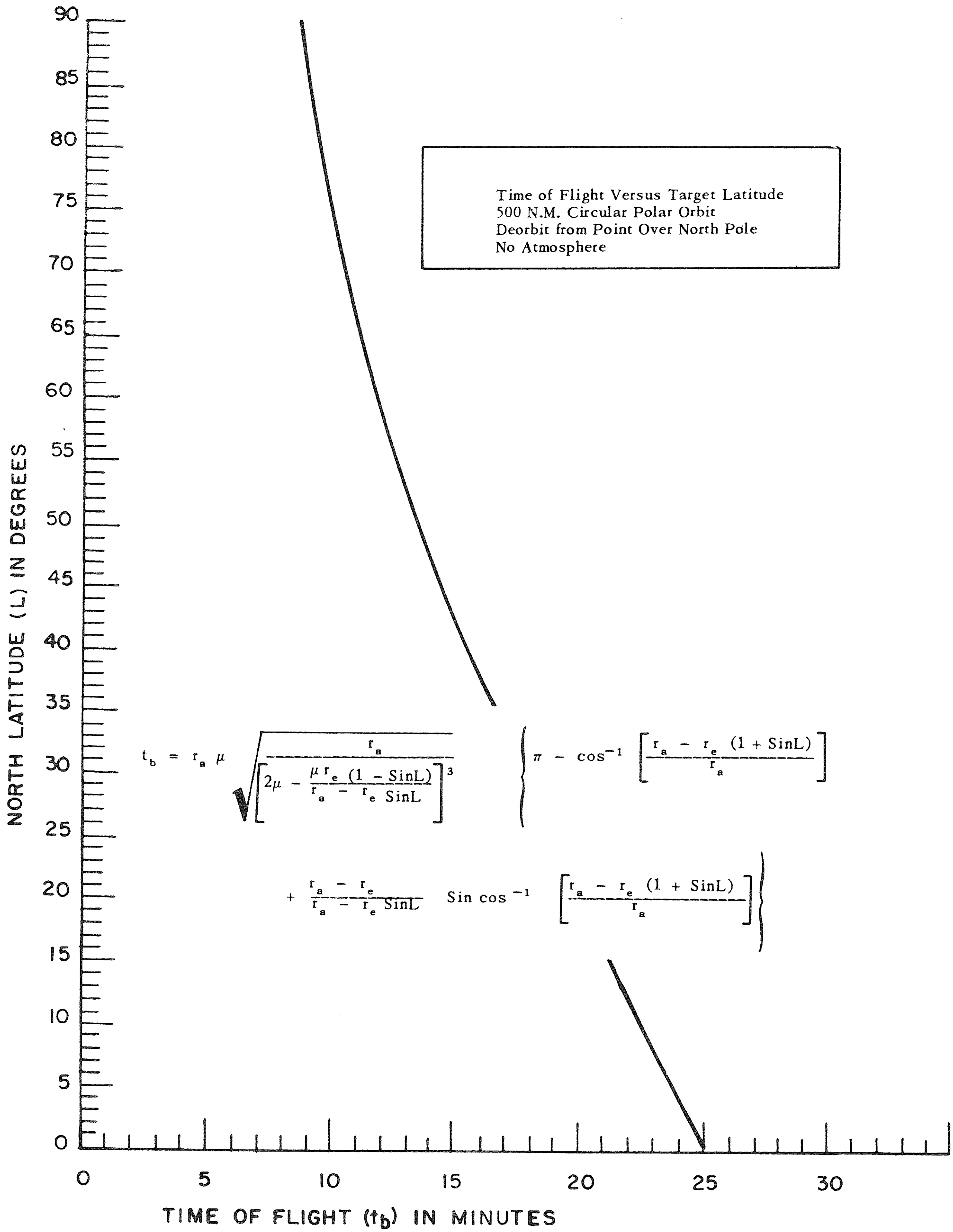


Figure 55



$$t_b = r_a \, \mu \sqrt{\frac{r_a}{\left[2\mu - \frac{\mu \, r_e \, (1 - \mathrm{Sin}L)}{r_a - r_e \, \mathrm{Sin}L}\right]^3}} \left\{ \pi - \cos^{-1}\left[\frac{r_a - r_e \, (1 + \mathrm{Sin}L)}{r_a}\right] + \frac{r_a - r_e}{r_a - r_e \, \mathrm{Sin}L} \, \mathrm{Sin} \cos^{-1}\left[\frac{r_a - r_e \, (1 + \mathrm{Sin}L)}{r_a}\right] \right\}$$

Figure 56

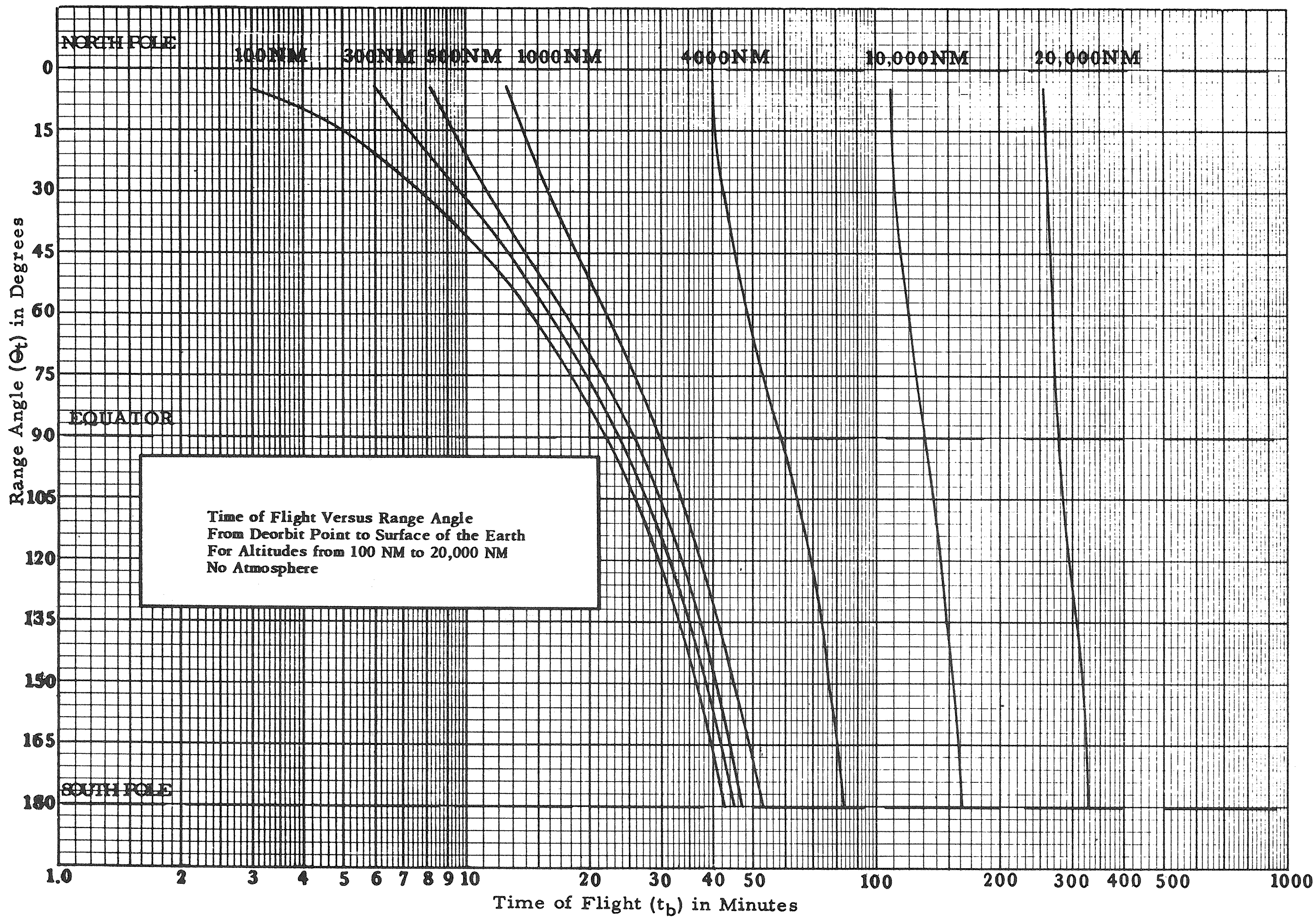

NORTH POLE
EQUATOR
SOUTH POLE
100NM
300NM
500NM
1000NM
4000NM
10,000NM
20,000NM
Time of Flight Versus Range Angle
From Deorbit Point to Surface of the Earth
For Altitudes from 100 NM to 20,000 NM
No Atmosphere
Range Angle (Θ_t) in Degrees
0
15
30
45
60
75
90
105
120
135
150
165
180
1.0
2
3
4
5
6
7
8
9
10
20
30
40
50
100
200
300
400
500
1000
Time of Flight (t_b) in Minutes

Figure 57

EQUATIONS PERTAINING TO BODIES IN MOTION

Linear Motion

1. (a) $v_{av} = \dfrac{s_f - s_o}{t_f - t_o} = \dfrac{\Delta s}{\Delta t}$

2. (a) $a_{av} = \dfrac{v_f - v_o}{t_f - t_o} = \dfrac{\Delta v}{\Delta t}$

3. (a) $s = v_o t + \dfrac{at^2}{2}$

4. (a) $v_f = v_o + at$

5. (a) $2as = v_f^2 - v_o^2$

(3–5: for uniform linear acceleration)

Angular Motion

1. (b) $\omega_{av} = \dfrac{\theta_f - \theta_o}{t_f - t_o} = \dfrac{\Delta \theta}{\Delta t}$

2. (b) $\alpha_{av} = \dfrac{\omega_f - \omega_o}{t_f - t_o} = \dfrac{\Delta \omega}{\Delta t}$

3. (b) $\theta = \omega_o t + \dfrac{\alpha t^2}{2}$

4. (b) $\omega_f = \omega_o + \alpha t$

5. (b) $2\alpha\theta = \omega_f^2 - \omega_o^2$

(3–5: for uniform angular acceleration)

Conversions from Angular Motion to Linear Motion

6. (a) $s = r\theta$

   (b) $v_t = r\omega$

   (c) $a_t = r\alpha$

7. $v_t = \dfrac{2\pi r}{P}$

8. $a_r = \dfrac{v_t^2}{r}$

Symbols

a—linear acceleration
P—period
r—radius
s—linear displacement
t—time interval
v—linear velocity
α—angular acceleration
θ—angular displacement
ω—angular velocity

Subscripts

av—average
f—final value
o—original value
r—radial
t—tangential

SOME USEFUL EQUATIONS OF ORBITAL MECHANICS

1. Eccentricity: $\epsilon = \dfrac{c}{a} = \dfrac{r_a - r_p}{r_a + r_p}$

2. Ellipse:
$$r_a + r_p = 2a$$
$$a - c = r_p$$
$$a + c = r_a$$
$$a^2 = b^2 + c^2$$

3. Specific Mechanical Energy:

$$E = \frac{v^2}{2} - \frac{\mu}{r}$$

$$E = -\frac{\mu}{2a}$$

4. Specific Angular Momentum:

$$H = v\,r\cos\phi$$

5. Two Body Relationships:

$$v_{circle} = \sqrt{\frac{\mu}{r}}$$

$$v_{ellipse} = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}}$$

$$v_{escape} = \sqrt{\frac{2\mu}{r}} = \sqrt{2gr}$$

$$P = \frac{2\pi a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}} = \left(5.30 \times 10^{-8}\,\frac{sec}{ft^{\frac{3}{2}}}\right)(a)^{\frac{3}{2}}$$

$$P^2 = \frac{4\pi^2 a^3}{\mu} = \left(2.805 \times 10^{-15}\,\frac{sec^2}{ft^3}\right)(a)^3$$

6. Variation of g:

$$g = \frac{\mu}{r^2}$$

7. Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

8. Constants:

$$r_e = 20.9 \times 10^6\ ft$$
$$r_e = 3440\ NM$$
$$\mu = 14.08 \times 10^{15}\ \frac{ft^3}{sec^2} \quad \text{For earth.}$$

$$1\ NM = 6080\ ft$$
$$\pi\ \text{radians} = 180°$$
$$1\ \text{radian} = 57.3°$$

9. Inclination of Orbital Plane

$$\cos i = \cos Lat \sin Azimuth$$

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